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The quest for truth, particularly in mechanics

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Abstract. The author ponders about things that necessarily come into engineering mind when the results obtained by theoretical, numerical and experimental approaches in solid continuum mechanics are correlated and compared with a pious wish to ascertain which of them are 'truer' or closer to 'reality'. This invokes many questions. How ancient and contemporary philosophers viewed the truth? How is truth related to consistency and validity of theoretical, numerical and experimental models we are inventing and employing? What is the role of threshold in physics, engineering, computation and in experiment? How are the basic quantities like time, force, stress, etc. defined? Do we properly understand them? What is the role of singularity in mathematics, physics and in engineering? The doubts stemming from uneasy answers to above pertinent questions are complemented by discussing examples from theoretical, numerical and experimental results obtained by solving dynamical problems in solid continuum mechanics. It should be stressed out that the role of doubts in our understanding the World plays a significantly positive role.

Key words: continuum mechanics, finite element analysis, validity of models, singularity, positive role of doubts.

Science cannot solve the ultimate history of nature. And that is because, in the last analysis, we ourselves are part of the mystery that we are trying to solve.

Max Planck

1. INTRODUCTION

When trying to answer the question what is a true approach to modelling processes in physics and engineering we have to start inquiring about Truth, about the models of Nature as well as about the nature of models.

Thomas Aquinas (1225–1274) claimed that the truth is an agreement of reality with perception. Today, however, the perceived reality depends on observation tools being used. For example, one could compare the results of observation obtained by magnifying glass with those of an electron microscope.

Immanuel Kant (1724–1804) asked for a clear distinction between the 'true reality' and 'perceived reality'. Kant argues that *in principle it is impossible to observe and study the world without disturbing it*. His ideas are very close to those of the Heisenberg principle of uncertainty.

2. 'TRUE' MODELS OF NATURE

As generally accepted today, the model is a purposefully simplified concept of a studied phenomenon, invented with the intention to predict – what would happen if .... Accepted assumptions (simplifications) consequently specify the validity limits of the model and in this respect the model is neither true nor false. The model, regardless of being simple or complicated, is good, if it is approved by an appropriate experiment [1].

When we, engineers, are modelling a particular phenomena of Mother Nature, the question of truth becomes irrelevant since the models we are designing, checking and using, either work or do not work to our satisfaction. So in this respect the mechanical theories, principles, laws and models, used in engineering practice, cannot be proclaimed true or false. They are either right or wrong. Furthermore, the right theories might fail when applied out of the limits of their applicability. A few examples might illustrate the previous claims.

- 1D wave equation is not able to predict stress wave pattern in a 3D body, and still is internally consistent and not wrong.
- Bernoulli–Navier slender beam theory 'fails' for thick beams.
- Newton's second law 'fails' for motion of bodies approaching the speed of light, and still it represents a perfect tool for engineering mechanics, including the computations and perfect prediction of celestial trajectories.
- Einstein's theory of relativity 'fails' when applied to quantum microcosms.

So it is obvious that we rather strive for robust models with precisely specified limits of validity and not for philosophically defined categories of truth and falsehood. From it follows that it is the validity of models, theories and laws that is of primary importance. How do we proceed in mechanical engineering?

- When trying to reveal the 'true' behaviour of a mechanical system, we are using an experiment.
- When trying to predict the 'true' behaviour of a mechanical system, we are accepting a certain theoretical model and then solve it analytically and/or numerically.

The trouble is that physical laws (or the models based upon them) cannot – in mathematical sense – be proved. We cannot, for example, prove the Newton's second law. But the Pythagorean theorem, as shown in Fig. 1, can be proved rather easily.

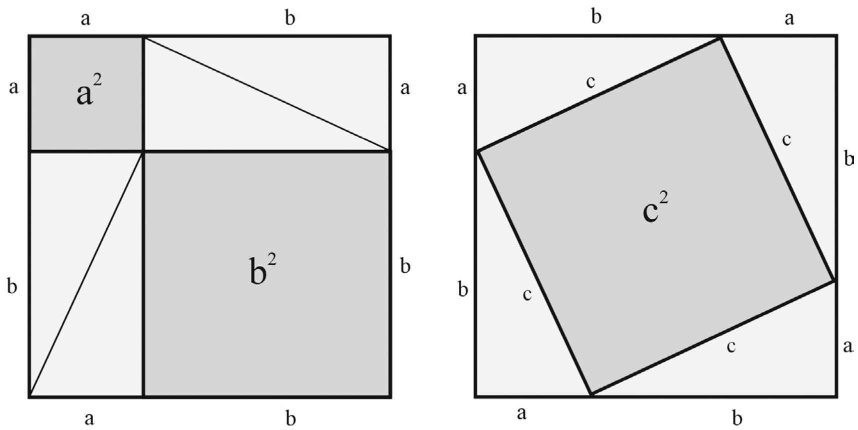

Fig. 1. Geometrical proof of the Pythagorean theorem.

And still one intuitively feels that a theorem is yet a less heavy-artillery term than a law. The terms, as law, theory, hypothesis, theorem, are not uniquely defined. “Words, words, words”[1].

3. MODEL VS. EXPERIMENT

To get rid of doubts we often claim that it is the experiment, which ultimately confirms the model in question. But experiments, as well as the subsequent numerical treatment of models describing the nature, have their observational thresholds. And sometimes, the computational threshold of computational analysis is narrower than that of the experiment. From this point of view a particular experiment is a model of nature as well.

Another mental hindrance we might have in our incessant quest for truth, is the lack of precise definitions of certain mechanical quantities. Definitions of conceptually defined quantities as force, stress, energy, etc are rather intuitive and often circular. A few examples from standard textbooks are the following.

Forces *are vector quantities which are best described by intuitive concepts such as push or pull* [2].

Force *is only a name for the product of acceleration by mass.* Attributed to d’Alembert and cited in ([1], p. 532).

Similarly definitions may be found for time. Intuitively, everybody knows what it is until the moment when a direct and precise definition is required.

In Book 11 of Confessions [3], St. Augustine ruminates on the nature of time, asking: *Quid est ergo tempus? Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio*. In medieval English it reads: *What then is time? If no one asks me, I know: If I wish to explain it to one that asketh, I know not.*

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1 Polonius: “What do you read, my lord?” Hamlet: “Words, words, words”. From Hamlet, Scene II. A room in the castle.

Other widely used variables as stress, energy, etc. may generate similar questions.

We have to emphasize, however, that these doubts, lack of precise definitions and inability to mathematically prove the mechanical laws, do not preclude our positive attitude to problem solving. It is often claimed that it is the experiment that is the only and ultimate judgment of the validity of the theory, model or a hypothesis being used. Is that really so? A few contradicting quotations might temporarily obscure our otherwise clear reasoning.

- *Experimental science does not receive truth from superior science. She is the mistress and the other sciences are her servants* (Roger Bacon: On Experimental Science, 1268).
- *I hope I shall not shock the experimental physicists too much if I add that it is also a good rule not to put overmuch confidence in the observational results that are put forward until they have been confirmed by theory*. Attributed to Eddington [4].
- *Experiment, indeed, is a necessary adjunct to a physical theory; but it is an adjunct, not the master*. See Truesdell and Toupin ([1], p. 227).

So far, it is 2 to 1 against the experiment – but it means almost nothing. One can find as many citations for and against as one wishes. Let us make it 3 to 1 by quoting Steve Hawking who – in author's view – seems to see it in proper relations. *Many, otherwise elegant and beautiful theories were rejected because they had not agreed with experimental observation – I do not know, however, of any big theory having been created as a direct generalization of an experiment* [5].

3.1. Our engineering goals: ability to explain and predict

For this we have many useful tools, namely theories, laws, models, computation, and last but not least the experiment. Let us have a closer look and start with thinking about models of time and space. Among many, there are two best known models explaining the origins of time and space: namely the Biblical and Big Bang models. Although our knowledge follows now the Big Bang model, the early philosophers who tried to understand nature and the essence of models, followed the Bible. And we all know that the explanatory power of mental tools is influenced by history. That is why it is of interest to start with the Biblical model.

3.2. Biblical model of time and space

Creation of the world according to Bible is described in the Book of Genesis. In the wording of King James Bible [6], it reads:

1:1 – In the beginning God created the heaven and the earth.

1:2 – And the earth was without form, and void; and darkness was upon the face of the deep. And the Spirit of God moved upon the face of the waters.

1:3 – And God said, Let there be light: and there was light.

1:4 – And God saw the light, and it was good: and God divided the light from the darkness.

So the Biblical timeline of the evolution is squeezed within six days: 1) God makes a firmament[2]; 2) the sea from the land is divided; 3) light appears; 4) God marks days, seasons and years; 5) he creates birds and sea creatures; 6) he makes wild beasts, livestock and reptiles. And finally God creates man in his own image.

Initial conditions of the biblical model – estimated age of the Earth varied violently throughout the history:

- 4004 BC, by Archbishop James Ussher in 1650;
- 6000 years ago, by Martin Luther;
- between 22 and 18 million years, by Hermann von Helmholtz in 1856;
- between 20 million and 400 million years, by Lord Kelvin in 1862;
- …
- today's estimate of the age of the Earth is 4.54×10^9 years.

3.3. Early doubts

Augustine of Hippo (354–430), also known as St. Augustine, emphasized that the text of the Bible was difficult to understand and should be reinterpreted as new knowledge became available. In particular, Christians should not make absurd dogmatic interpretations of scripture which contradict to what people know from physical evidence.

In the 13th century, **Thomas Aquinas** cautioned that since Holy Scripture can be explained in a multiplicity of senses, one should not adhere to a particular explanation, only in such measure as to be ready to abandon if it be proved with certainty to be false.

Their ideas stood the test of time. In fact, they constitute the very basic principles of today's scientific methods.

3.4. Big Bang model – the timeline of the evolution

The Universe was born at a finite time in the past. At the beginning the Universe was concentrated into a single point, where and when the fabric of time and space came into existence. The Universe was filled homogeneously and isotropically with an infinitely high energy, density, temperature and pressure.

The earliest period of time in the history of the universe, from zero to approximately 10^{-43} seconds, is defined as the Planck time. At that time the Universe occupied the space, whose dimension is expressed by so-called Planck length (the distance light travels in one Planck time unit) – of about 1.616×10^{-35} metres. During early history of the Universe, the physical characteristics such as mass, charge, flavour and colour charge were meaningless. The boundary conditions are **singular**. It is not known whether something existed before that singularity. Some authors even say that the question – what existed before that singularity – is meaningless. Since the Big Bang represents the birth of time and space, no time could have existed before.

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[2] In biblical terminology the firmament is a great tent-like ceiling made of solid crystalline material.

In this respect the Biblical model could at least answer the annoying question related to what was before the Creation itself – it was God, being eternal, immortal, omnipotent and omniscient, who might have existed.

Today it is understood that the estimated age of the Universe is 13.75×10^9 years. For a detailed account of events occurring during the first three minutes after the Big Bang see Weinberg's book [7]. What might happen within the last three minutes before the Big Crunch is vividly described by Davies in [8]. For about what might happen after the Big Crunch see Penrose's book [9].

3.5. Contemporary doubts

The Universe had a beginning, which is philosophically troubling and physically singular. It was created out of nothing and concentrated into a single point with infinite temperature, density, etc. The Big Bang theory *does not* provide any explanation for such an initial condition; rather, it *describes* and *explains* the general evolution of the universe going forward from that point on. How closely we can extrapolate towards the singularity is debated – certainly no closer than the end of the Planck epoch.

4. SINGULARITIES

4.1. Absolute singularity

Georg Cantor (1845–1918) coined the term *absolute infinity* for the totality of everything that is something beyond the mathematical description of representation which could only be comprehended by the mind of God. Closely connected to absolute infinity are questions related to what was before God created the world.

4.2. Mathematical singularity

Mathematical singularity is a mental invention and could only happen in our minds. Still, it is a standard part of mathematical analysis as in $\lim_{x \to 0} \frac{1}{x} \to \infty$.

A strange character of infinity is nicely documented by an amusing story about the Infinity Hotel, which is attributed to David Hilbert (1862–1943):

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The Infinity Hotel has infinitely many rooms.
Imagine that one evening all the rooms, numbered 1, 2, 3, etc., are
occupied.
There comes a new guest to the reception asking: Do you have a room for
tonight?
No problem, says the receptionist, starts his notebook and invokes a simple
procedure.

i = 1;
Until <all the guest are displaced> do
      Move the guest from room (i) to room (i+1);
      i = i + 1;
End of do

This way, the newly arrived guest will get the room No. 1.
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Actually, any countable number of guests can be accommodated this way. That is the infinite number of buses, each carrying an infinite number of guests. Logically there is no flaw in the story. Practically, the process would require the infinite time and the infinite amount of energy.

4.3. Physical singularities or rather singularities appearing in mathematical models describing physical phenomena

Out of many, let us cite a few examples:

- infinite displacement, strain and stress under the point force in solid continuum mechanics;
- infinitely fast shock wave change of pressure accompanying sonic boom in fluid mechanics;
- infinite stress at the crack tip in fracture mechanics models.

Philosophers, mathematicians and physicists had different views about the existence of infinity. In Rucker's book [10] one can find the following table. Of course, '1' is a positive attitude to a particular type of singularity.

	Infinity		
	Mathematical	Physical	Absolute
Abraham Robinson	0	0	0
Thomas Aquinas	0	0	1
Plato	0	1	0
Luitzen Brouwer	0	1	1
David Hilbert	1	0	0
Kurt Goedel	1	0	1
Bertrand Russell	1	1	0
Georg Cantor	1	1	1

Intuitively we feel that a singularity, appearing in a physical model, always means a warning concerning the range of validity of that model. Usually, a more general model – having a wider scope of validity – is invented with the intention to remove that singularity. And very often there is no need to discard the older and simpler model, since it might be perfectly useful in the validity range for which it was originally conceived.

Physicists in their statements express rather strong views on singularity – it could be documented by two quotations:

- *A singularity brings about so much arbitrariness into the theory that it actually nullifies its laws* (from Einstein and Rosen [11]).
- *… a theory that involves singularities carries within itself the seed of its own destruction* (from Bergmann [12]).

Singularity, they claim, signals the breakdown of the model. Engineering views, however, on singularity are not so strong.

- Appearance of singularities in equations describing the behaviour of mechanical quantities in mathematical models of nature signals that the particular model in question is incomplete.
- Appearance of singularity in a model merely suggests that the theory being employed has reached the limits of its validity and must be superseded by new and improved version, which should replace the singularity by a finite quantity.

5. A FEW EXAMPLES OF SINGULARITIES APPEARING IN ENGINEERING MODELS

5.1. Transient loading of elastic half-space by a point force, whose time distribution is given by the Heaviside function

The original Lamb's analytical analysis, as cited in [13], employs Fourier' superposition of harmonic waves for the transient normal loading on a half-space. Solving the Lamb's problem on a free boundary is relatively simple and is available in a closed form, representing the distributions of radial and axial displacements, as functions of time and space. The solution is attributed to Pekeris ([14], p. 368, Eq. (6.3.150)), whose analytical formulae are easy to evaluate.

Computed axial surface displacements, depicted in Fig. 2 in space-time coordinates, show two significant singularities occurring there as a direct consequence of point force loading. Also the arrival of longitudinal, shear and Rayleigh waves can be observed.

As a rule, the available analytical solutions are often based on a point force loading whose time dependencies are usually prescribed by Heaviside or Dirac functions. This fact partially simplifies the dreary workload needed for carrying out the analytical analysis.

For the purposes of finite element (FE) analysis, the half-space was modelled by axisymmetric elements (bilinear and quadratic shape functions for coarse, medium and fine meshes) forming a cylinder, whose symmetrical part is sketched in Fig. 3. The time distribution of the loading point force, applied at point A, is prescribed by the rectangular pulse, whose duration corresponds to time when the primary longitudinal wave reaches half the length of the specimen.

Until the primary longitudinal wave reaches the surface of the cylinder and/or the face opposite to the loaded one, the considered cylinder is a proper representation of the half-space.

The finite element results of the Lamb's problem demonstrate a sort of 'divergence' behaviour as a function of increasing mesh density. The response depends also on whether the element shape function is linear or quadratic. This is shown in Fig. 4, where the time dependence of axial displacements directly under

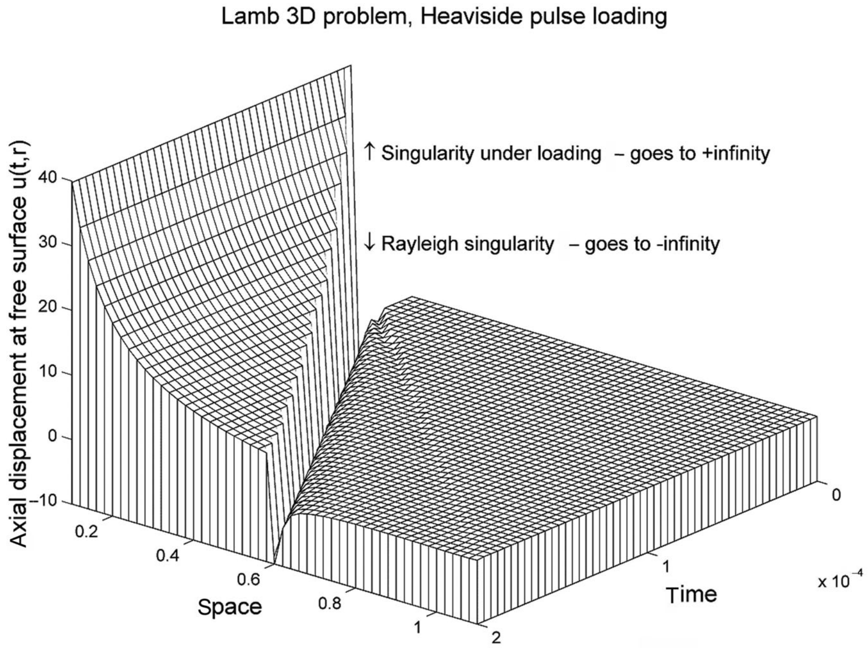

Fig. 2. Time-space representation of axial displacement on the surface of half-space.

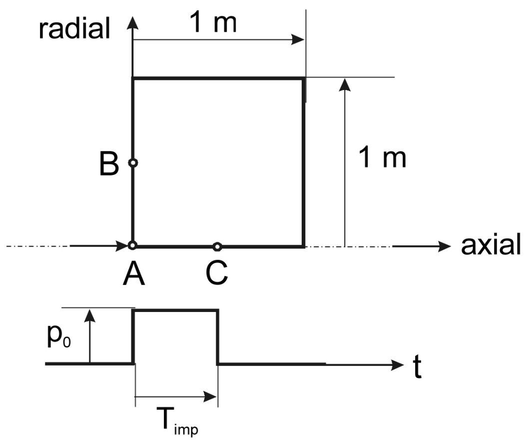

Fig. 3. FE analysis – half-space treated as axisymmetric space.

the point loading force is depicted. The finer meshes yield gradually greater displacements under the loading point force.

Figure 4 also shows the influence of the element type used for finite element modelling. One can also see that the double mesh density of bilinear elements (L)

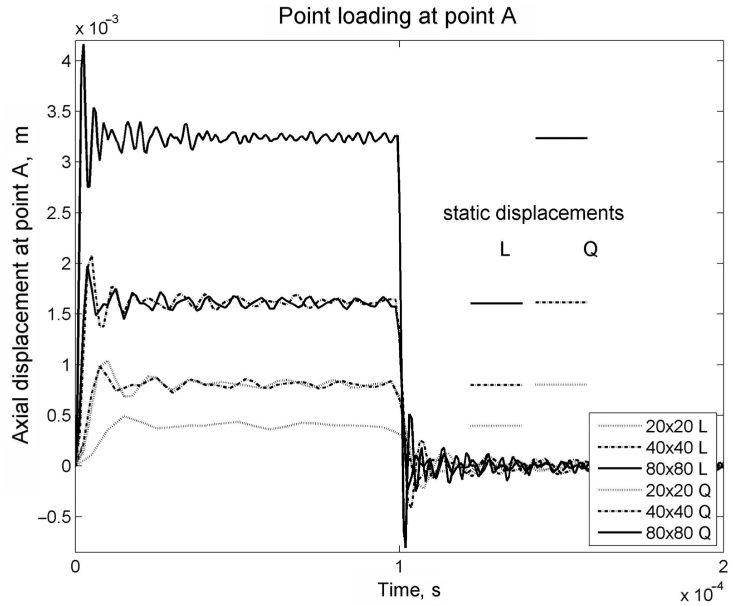

Fig. 4. Axial displacements under the point loading force increase with decreasing mesh size.

produces roughly the same results as the reference density with biquadratic elements (Q). The employed meshes are 20 by 20, 40 by 40 and 80 by 80, respectively for the considered space domain. Bilinear and biquadratic square axisymmetric elements, full quadrature, consistent mass matrix formulation and Newmark time step operator without algorithmic damping were systematically used. For more details see [15].

One can conclude that the finite element method – being an orthodox daughter of continuum mechanics, where the notion of a point force is forbidden since it leads to singularity – would give infinite displacements for infinitely fine mesh.

The results of a similar numerical experiment, this time with a mesh-dependent equivalent pressure loading, are shown in Fig. 5. Of course, the singularity phenomenon disappeared.

In solid continuum, the effects of different localized equivalent loads cannot be distinguished in areas located sufficiently far from their applications. This is what St. Venant's principle states. In discretized continuum, however, the effects of point and equivalent distributed loadings are different and furthermore mesh dependent. This is manifested by a paradoxical increase of stored potential and kinetic energies with the increasing mesh density, which is plotted in Fig. 6 as a function of time. One is almost tempted to employ this phenomenon for a pollution-free energy production. Unfortunately, the pollution-free energy production problem disappears when an equivalent distributed (non-point) loading is applied [16].

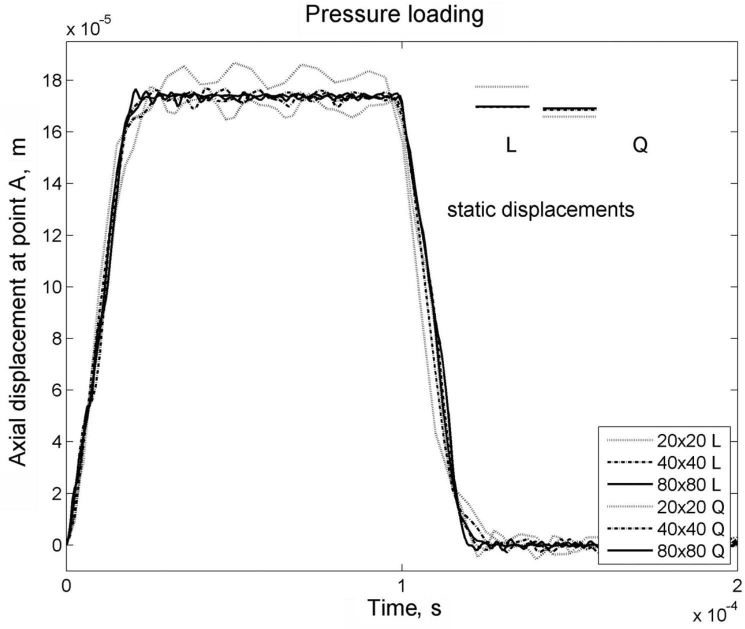

Fig. 5. Axial displacements under the pressure loading almost do not depend on mesh size.

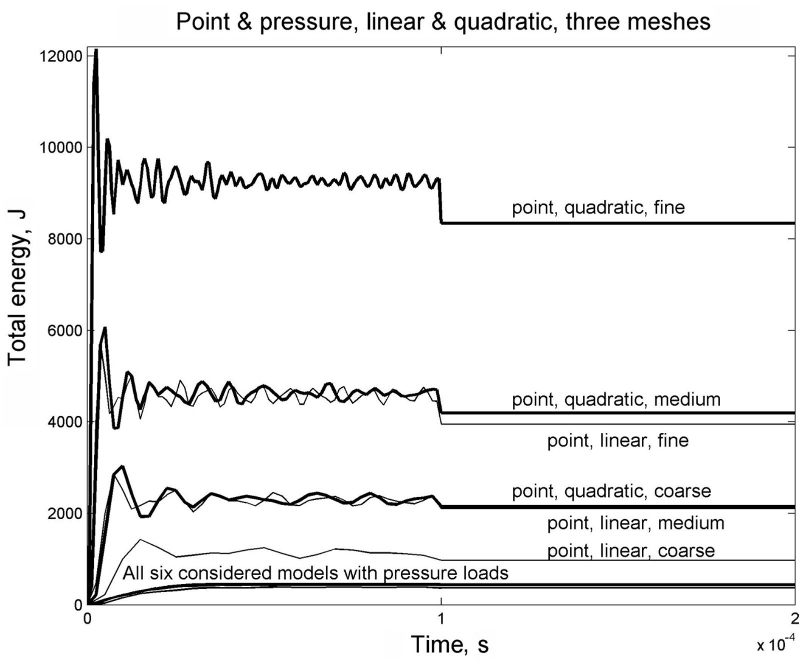

Fig. 6. Pollution-free energy production.

Seemingly unproblematic model of elastic continuum has embedded singularities in it. For example, a point force, a frequently used tool in engineering analysis, is a forbidden entity in continuum mechanics since it leads to a singularity response – this is manifested by the fact that the displacements, strains and stresses under the application of a point force tend towards infinity.

To a certain extent this property is retained when the continuum is treated by means of a FE model. Actually, it is smeared out by the existence of shape functions but with diminishing mesh size, but it is still manifested by a significant increase of displacement under the application of a point (nodal) force.

The FE mesh made of 'null-sized' elements would provide the infinite displacement under the application of a nodal force as the continuum model. So making a finer and finer mesh we are representing better and better those continuum properties that are mathematically correct but physically unattainable.

5.2. 'Infinite' speed of propagation in FE analysis using Newmark time integration operator

Several time marching operators for solving the systems of ordinary differential equations, suitable for the FE modelling of transient tasks of solid continuum mechanics, are known today. The detailed description of their background and analysis of their properties can be found, e.g., in [17–19]. Commercial FE packages offer a plethora of approaches [20,21]. The outlines and rules for their 'safe' usage are generally advocated; nevertheless it still might be of interest to analyse in detail the minute differences, obtained by applying two different integration methods to the same task.

Let us concentrate our attention to the comparison of results obtained by Newmark (NM) and central difference (CD) methods when a transient task in solid continuum mechanics is solved.

The NM method is a classical representative of implicit methods. Used with consistent mass matrix and without algorithmic damping it conserves energy and is unconditionally stable. In order to minimize the temporal and spatial discretization errors, the NM method is recommended [17], to be used with consistent mass matrix formulation.

The CD method, a representative of explicit methods, is only conditionally stable. When used within its stability limits with consistent mass matrix formulation it also fully conserves energy. To reduce the temporal and spatial discretization errors the CD method is recommended [17], to be used with diagonal (lumped) mass matrix formulation. Using it with a consistent mass matrix is possible but practically prohibitive for two reasons. First, the problem becomes computationally coupled – the individual differential equations cannot be solved independently. Second, the data storage demands for the consistent mass matrix are substantially higher than those needed for a diagonal mass matrix. Today, the CD method is almost exclusively used with the diagonal mass matrix formulation, which is furthermore plausible from the point of view of minimization of dispersion effects. But using the CD method with diagonal mass matrix we are punished a little bit by the fact that the time dependence of total mechanical energy slightly fluctuates around its 'correct' value [22].

Comparison of the computed time history of axial strains on the surface of a cylindrical tube at a location, whose distance from the impacted face of the tube is known, obtained by NM and CD methods using 3D elements, is presented in Fig. 7. Eight-node brick element and the same time integration step (1×10^{-7} s) were used in both cases. The proper choice of the time step value is discussed in [23]. For the NM method the consistent mass matrix was employed, while the diagonal mass matrix was used for the CD method. For more details see [22,24].

The left-hand subplot of Fig. 7 presents the axial strains as functions of time steps in the above mentioned location. The negative peak, denoted IL1, corresponds to the immediate position of the incoming loading pulse. There is a visible difference between NM and CD results, which – from the engineering point of view – seems to be small. Often, the differences are viewed by the prism of the plotting scale.

In the upper right-hand subplot of Fig. 7, which is the enlarged view of the small rectangle, presented on the left-hand side of Fig. 7, the theoretical positions of arrivals of hypothetical 3D (c_L) and 1D (c_0) longitudinal waves are indicated by vertical lines. Of course, in a bounded 3D body no pulse, being composed of infinitely many harmonics, does propagate by any of above mentioned velocities. But the theoretical wave speeds are useful bounds for our expectations. The detailed strain distributions, obtained by NM and CD methods, are shown as

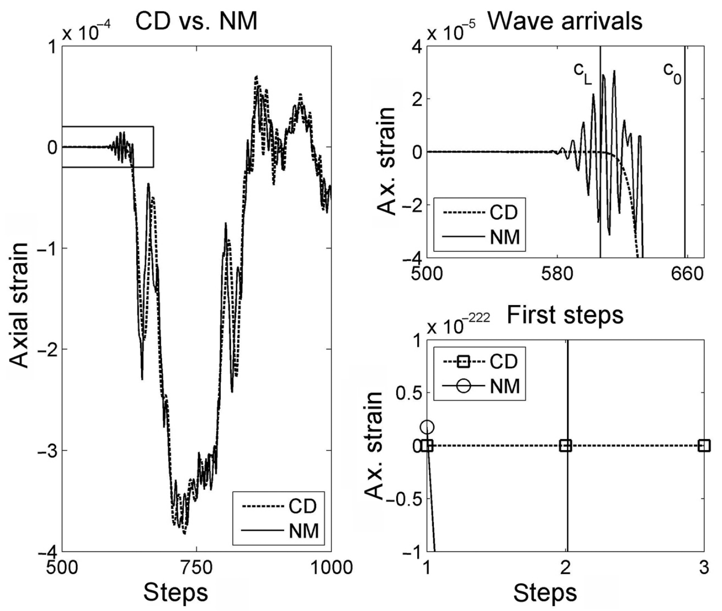

Fig. 7. Time distributions of surface axial strains obtained by NM and CD operators.

well. From the analysis of dispersion properties of finite elements and that of time integration methods, presented in detail in [17], it is known that the computed speed of wave propagation for the CD approach with diagonal mass matrix underestimates the actual speed, while using the NM approach with consistent mass matrix the actual speed is overestimated.

Less known is the fact that the speed of propagation, modelled by NM method with consistent mass matrix formulation, is actually ‘infinitely’ large – meaning that at the end of the first integration time step the most distant element ‘knows’ that the modelled mechanical system was loaded. A brief explanation of this curiosity could be sketched as follows.

5.3. Interlude – assessment of ‘variable computational speeds’ of wave propagation by analysing two time marching algorithms for the numerical integration of the system of ordinary differential equations $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$

The central difference (CD – left column) method and the Newmark (NM – right column) method lead (in principle) to the repeated solutions of the system of algebraic equations at each time step

$$\frac{1}{\Delta t^2}\mathbf{M}\mathbf{q}_{t+\Delta t} = \tilde{\mathbf{P}}_t, \tag{1}$$

$$\hat{\mathbf{K}}\mathbf{q}_{t+\Delta t} = \hat{\mathbf{P}}_{t+\Delta t}, \tag{2}$$

where Δt is time step, $\mathbf{q}_{t+\Delta t}$ are unknown displacements at time $t+\Delta t$ and $\mathbf{M}$ is the mass matrix. The effective loading forces and the effective stiffness matrix are

$$\tilde{\mathbf{P}}_t = \mathbf{P}_t - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right)\mathbf{q}_t - \frac{1}{\Delta t^2}\mathbf{M}\mathbf{q}_{t-\Delta t}, \tag{3}$$

$$\hat{\mathbf{P}}_{t+\Delta t} = \mathbf{P}_{t+\Delta t} + \mathbf{M}(c_1\mathbf{q}_t + c_2\dot{\mathbf{q}}_t + c_3\ddot{\mathbf{q}}_t), \tag{4}$$

$$\hat{\mathbf{K}} = \mathbf{K} + \frac{1}{\beta\,\Delta t^2}\mathbf{M}, \tag{5}$$

where $\mathbf{K}$ is the stiffness matrix. Time derivatives are denoted by dots. Lower indices indicate time. Definition of constants appearing above and more details are given in [18].

Generally, the matrices $\mathbf{K}, \mathbf{M}, \hat{\mathbf{K}}$ are sparse and banded. Nevertheless, their inversions[3] $\mathbf{K}^{-1}, \mathbf{M}^{-1}$ as well as $\hat{\mathbf{K}}^{-1}$ (needed for extracting the displacements

[3] Of course, the actual inversion is never carried out; the system of algebraic equation is solved instead.

$\mathbf{q}_{t+\Delta t}$ at the next time step from equations (1) and (2)) are full. From it follows that in both systems of equations the unknowns are coupled. This means that when calculating the *i*-th displacement, there are all other displacements, which – through the non-zero coefficients of a proper inverse matrix – are contributing to it.

Thus, when (at the beginning of the integration) a nonzero loading is applied at a certain node, then (at the end of the first integration step) the displacements at all nodes of the modelled mechanical system are non-zero, indicating that the whole system already ‘knows’ that it was loaded, regardless of the distance between the loading node and the node of interest.

The magic spell could only be broken if the matrix, appearing in the system of algebraic equations, is diagonal, because its inversion is then diagonal as well. This, however, could only be provided for the CD approach, operating with mass matrix, because it is only the mass matrix which can be meaningfully diagonalized (see [[25]]).

End of interlude.

The above discussion is illustrated in the lower right-hand side subplot of Fig. 7, where one can see the strains computed by CD and NM operators (at a finite location from the loading area) during the first three steps of integration. The CD operator, with a diagonal mass matrix, gives the expected series of pure zeros, while the NM method gives values negligibly small (of the order of 10^{-222}) but still non-zero. It should be emphasized that this has nothing to do with round-off errors. The same phenomenon would have appeared even if we had worked with symbolic (infinitely precise) arithmetics.

6. THRESHOLDS

6.1. Computational and observational thresholds

The *computational threshold* depends on the number of significant digits used for the mantissa representation of the floating number [[13]].

The minimum floating point number that can be represented by the standard double precision format (which is a default today) is of the order of 10^{-308}. This is our numerical observational threshold allowing distinguishing the value 10^{-222} in the first step of the lower right-hand side of Fig. 7.

If, for the same numerical integration process in time, we had employed the single precision format (the threshold of the order of 10^{-79}), we would have observed pure zero in the first step instead and the first non-zero value would appear later.

Let us imagine that we would like to measure (experimentally) the wave speed by sitting at a certain observational node (whose distance from the loaded node is known) and measuring the time needed for the arrival of the ‘measurable’ or ‘detectable’ signal coming from the distant source of loading.

The measurable signal is such that, in absolute value, is greater than a 'reasonable' *observational threshold*. And what is a proper value of it is a good question.

A thought experiment accompanied by FE computation might help. Imagine a standard finite element double-precision computation giving at a certain time the spatial distribution of displacements at a node on the surface of a body. Assume that the distance of our observational node from the loading node is known. Now, let us set a 'reasonable' value of the threshold and apply a sort of numerical filter on obtained displacements, which erases all the data whose absolute values are less than the mentioned value of the threshold. This way, for a given threshold value, we get a certain arrival time and from the known distance we obtain the propagation speed. Working with displacements, normalized to their maximum values, allows us to consider the threshold values as the relative ones. For more details see [22].

Varying the simulated threshold value in the range from 10^{-6} to 10^{-1} we will get a set of different velocities of propagation. As a function of threshold they are plotted in Fig. 8. Material constants for the standard steel were used. The horizontal lines represent the theoretical speeds for longitudinal waves in the 3D continuum, for longitudinal plane stress waves in the 2D continuum as well as for the shear waves. Obviously, the shear wave speeds are identical both for 3D and 2D cases [13].

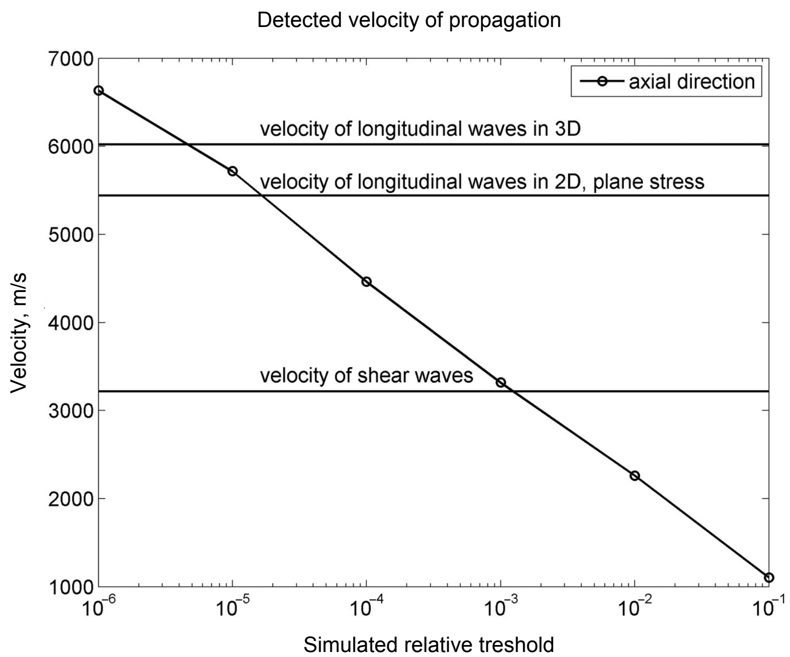

Fig. 8. Detected velocity of propagation vs. relative threshold.

The previous discussion might appear rather academic. The threshold issue, however, is really important when the speed of propagation is being determined by experimental means. The procedure is the same as in the numerical simulation approach. Observing the first 'measurable' response at a certain time in a given distance from the loading point, one can estimate the speed of propagation. As before, the estimated velocity value depends on the observational threshold value. There is, however, a significant difference. While we could almost arbitrarily vary the simulated threshold value in the numerical treatment, the value of observational threshold is usually constant for the considered experimental setup being used for the measurement of a particular physical quantity.

It is known that the longitudinal waves carry substantially less amount of energy than these of the shear and Rayleigh waves and that the surface response, measured in displacements or strains, is of substantially less magnitude for the former case.

From the experimental point of view one can conclude that for a correct capturing of the longitudinal velocity value, the relative precision of at least of the order of 10^{-6} is required. This is a tough request. The relative threshold of the order of 10^{-3} is more common in experimental practice. However, in an experiment with the relative precision of the order of 10^{-3}, one would not detect the arrival of longitudinal waves and might wrongly conclude that the first arriving waves are of the shear nature or would estimate the velocity of propagation of the order of 3000 m/s.

All this fuzz is about the margins of our ability to distinguish something against nothing. This is, however, crucial for any meaningful human activity.

6.2. Finite element and solid continuum threshold

The model of linear elastic continuum has embedded singularities in itself. Using the continuum model, the displacement under the application point of a concentrated force tends towards infinity. The FE mesh, made of null-sized elements, would provide the infinite displacement under the application point of a point force as the continuum model. So making a finer and finer mesh we are representing better and better those continuum properties that are physically unacceptable.

Modal properties of the continuum model pose another example of the continuum singularity. The continuum model has an unlimited spectrum of eigenfrequencies and an infinite number of modes of vibrations. The FE model has a finite number of eigenfrequencies and eigenmodes; furthermore, the higher ones are distorted by dispersion. The highest eigenfrequency of the FE model is related to dimensions of the smallest element appearing in the mesh while that of the continuum model tends to infinity. The period of the FE eigenmode, corresponding to the highest eigenfrequency, is the shortest one, but still finite, while that of the continuum model tends to zero. This is not, however, troubling us too much since the frequency, corresponding to an element of continuum of the size of inter-atomic distance (10^{-10} m for metals), is of the order of GHz.

The transient response of a linear structure could be obtained by modal superposition, i.e. as a superposition of all eigenmodes with amplitudes depending on initial and boundary conditions and on the Fourier spectrum of the loading pulse. Again, making a finer mesh we are getting better representation of those continuum properties that are physically ‘wrong’.

Fortunately, the nature is kind to us, since the energy is predominantly carried out by the lower modes of the spectrum.

Fast transient problems, however, contain high frequency harmonics, and in FE modelling they require small elements and a lot of eigenmodes to be taken into account. If step-by-step approach is used instead of modal superposition, a very short time step of integration should be employed. And this leads to a question. That is, up to which frequency limit is the FE approach trustworthy?

We know that the FE method is a model of continuum. The continuum – also a model – being based on the continuity hypothesis, disregards the corpuscular structure of matter. It is assumed that matter within the observed specimen is distributed continuously and its properties do not depend on the specimen size. Quantities describing the continuum behaviour are expressed as piecewise continuous functions of time and space. It is known [13] that such a conceived continuum has no upper frequency limit. To find a ‘meaningful’ frequency limit of the FE model, which is of discrete – not continuous – nature, one might pursue the following heuristic reasoning.

Imagine a uniform finite element mesh with a characteristic element size, say h. Trying to safely ‘grasp’ a harmonic component (having the wavelength λ) by this element size we require that at least five-element length fits the wavelength. This leads to $\lambda = 5h$. What is the frequency of this harmonics? Taking a typical wave speed value in steel of about $c = 5000$ m/s and realizing that $\lambda = cT$ and $f = 1/T$, we get the sought-after ‘frequency limit’ in the form $f = c/(5h)$. For a one-millimetre element we get $f = \frac{5000}{5\times 0.001} = 1\times 10^6\,\mathrm{Hz} = 1\,\mathrm{MHz}$. Let’s call it the *five-element frequency*, denoting it $f_{5\mathrm{elem}}$ in the text.

Figure 9, based on the above accepted five-element assumption, shows the relation between the element size in mm and ‘safely’ attainable frequency of a FE model measured in MHz. The plot is complemented by a typical austenite steel grain size indicating thus the validity limits of the continuum model as well. The existence of material grains is in strict disagreement with the concept of ‘continuity’, in which the properties of the model do not depend on the size of the observed specimen. Finally, there are three horizontal lines indicating 1 GHz, 100 MHz and 1 MHz levels. The first being out of reach of present day modelling, the second representing the maximum sampling limit of present day recording oscilloscopes and the last one the frequency level attainable with 1 mm elements.

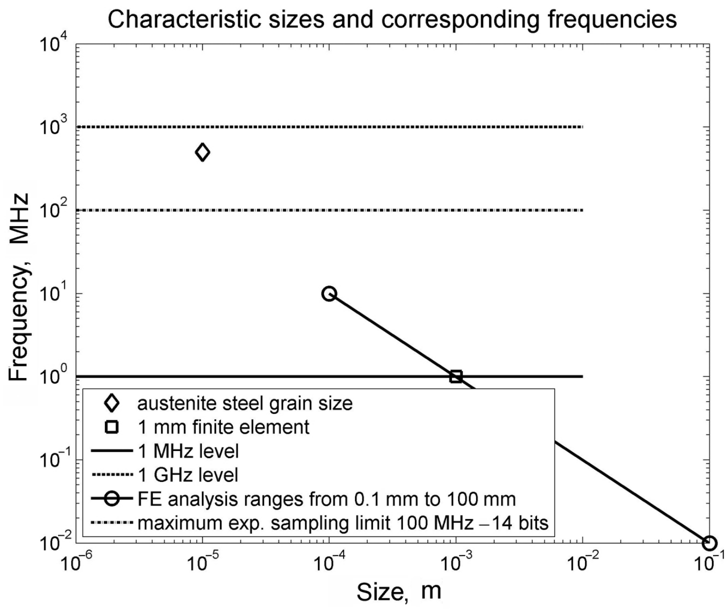

Fig. 9. Finite element mesh size vs. attainable frequency.

7. CONCLUSIONS

Mechanical theories, principles, laws and models, used in engineering practice, cannot be proclaimed true or false. They are either right (working to our satisfaction) or wrong. Regardless of being simple or complicated, they are 'right', if approved by an *appropriate* experiment (i.e. conceived in agreement with accepted assumptions of the theory). History reveals that wrong theories might appear, but not being confirmed by experiments, are quickly discarded as ether or flogiston. Theories are right only within the limits of their applicability. We cannot claim that a theory being proved by an experiment is right. The only thing we can safely state is that such a theory is not proved wrong.

Generally, a singularity appearing in a model always means a serious warning concerning the range of validity of that model. Usually, a more general model – having a wider scope of validity – is invented removing that singularity. Very often there is no need to discard the older and simpler model, since it might perfectly work in the validity range for which it was conceived. The role of doubts on our way to understanding the nature is far from negative.

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Tõeotsingul mehaanikas

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On käsitletud tahkisemehaanika probleeme tõeotsingu seisukohalt, küsides, millised teoreetilised, arvutuslikud või numbrilised tulemused on tõesed ehk lähemal reaalsusele. Vastused peavad põhinema meie mõttemaailma kujundanud filosoofide arutlustel. Nende alusel on vaadeldud näiteid tüüpilistest inseneriülesannetest: elastse poolruumi koormamine koondatud jõuga ja lainelevi silindrilises torus. Analüüs baseerub lõplike elementide meetodil. Järeldusena märgib autor, et teooriaid ja mudeleid mehaanikaprobleemide analüüsil tuleb hinnata eelkõige eksperimentidega sobivuse seisukohalt, mitte aga neid kas õigeteks või valedeks kuulutada.