Proceedings of the Estonian Academy of Sciences,
2008, **57**, 4, 205–209
doi: 10.3176/proc.2008.4.01
Available online at www.eap.ee/proceedings

MATHEMATICS

On a new application of power increasing sequences

Hüseyin Bor

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey; bor@erciyes.edu.tr

Received 8 May 2008, in revised form 13 August 2008

Abstract. A general theorem dealing with $\mid C,\alpha,\beta\mid_k$ summability factors has been proved under weaker conditions.

Key words: absolute summability, power increasing sequences, summability factors.

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the nth Cesàro means of order (α,β), with $\alpha+\beta>-1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [3])

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-v} A_v^{\beta} s_v,$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta})$, $\alpha+\beta>-1$, $A_0^{\alpha+\beta}=1$, and $A_{-n}^{\alpha+\beta}=0$ for $n>0$.

The series $\sum a_n$ is said to be summable $\mid C,\alpha,\beta\mid_k$, $k\geq 1$ and $\alpha+\beta>-1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k < \infty. \tag{2}$$

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [5]), condition (2) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{3}$$

If we take $\beta=0$, then $\mid C,\alpha,\beta\mid_k$ summability reduces to $\mid C,\alpha\mid_k$ (see [6]) summability. Bor and Srivastava [2] have proved the following theorem for $\mid C,\alpha\mid_k$ summability factors of infinite series.

Theorem A. *Let (X_n) be an almost increasing sequence and there be sequences (β_n) and (λ_n) such that*

$$| \Delta\lambda_n | \leq \beta_n, \tag{4}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{6}$$

$$| \lambda_n | X_n = O(1) \quad \text{as} \quad n \to \infty. \tag{7}$$

If the sequence (u_n^α), defined by (see [8])

$$u_n^\alpha = \begin{cases} | t_n^\alpha |, & \alpha = 1 \\ \max_{1\leq v\leq n} | t_v^\alpha |, & 0 < \alpha < 1 \end{cases}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n}(u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $| C,\alpha |_k$, $k \geq 1$ and $0 < \alpha \leq 1$.

2. THE MAIN RESULT

The aim of this paper is to generalize Theorem A under weaker conditions. For this we need the concept of quasi δ-power increasing sequence. A positive sequence (γ_n) is said to be a quasi δ-power increasing sequence if there exists a constant $K = K(\delta,\gamma) \geq 1$ such that

$$Kn^\delta \gamma_n \geq m^\delta \gamma_m \tag{8}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi δ-power increasing sequence for any nonnegative δ, but the converse is not true for $\delta > 0$. Moreover, for any positive δ there exists a quasi δ-power increasing sequence tending to infinity, but it is not almost increasing. In fact, if we take (γ_n) is an almost increasing, that is, if

$$Ac_n \leq \gamma_n \leq Bc_n$$

holds for all n with an increasing sequence (c_n), then for any $n \geq m \geq 1$

$$\gamma_m \leq Bc_m \leq Bc_n \leq \frac{B}{A}\gamma_n$$

also holds, whence (8) follows obviously for any $\delta \geq 0$ with $K = \frac{B}{A}$. Thus any almost increasing sequence is quasi δ-power increasing for any $\delta \geq 0$. We can show that the converse is not true. For this, if we take $\gamma_n = n^{-\delta}$ for $\delta > 0$, then $\gamma_n \to 0$. Thus it is obviously not an almost increasing sequence (see [7] for extra details).

We shall prove the following theorem.

Theorem. *Let (X_n) be a quasi δ-power increasing sequence for some $0 < \delta < 1$. If conditions (4)–(7) are satisfied and the sequence $(\theta_n^{\alpha,\beta})$ defined by*

$$\theta_n^{\alpha,\beta} = \begin{cases} | t_n^{\alpha,\beta} |, & \alpha = 1,\ \beta > -1 \\ \max_{1\leq v\leq n} | t_v^{\alpha,\beta} |, & 0 < \alpha < 1,\ \beta > -1 \end{cases}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n} (\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid C, \alpha, \beta \mid_k$ *for* $0 < \alpha \leq 1$, $\beta > -1$, $\alpha + \beta > 0$, *and* $k \geq 1$.

It should be noted that if we take $\beta = 0$ and (X_n) as an almost increasing sequence, then we get Theorem A.

We need the following lemmas to prove our theorem.

Lemma 1 ([7]). *Under the conditions on* (X_n), (β_n), *and* (λ_n) *as taken in the statement of the theorem, the following conditions hold*:

$$nX_n\beta_n = O(1),$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Lemma 2. *If* $0 < \alpha \leq 1$, $\beta > -1$, *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \right|.$$

The proof of Lemma 2 is similar to the proof of the Lemma of Bosanquet (see [4]) and we omit it.

3. PROOF OF THE THEOREM

Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by (1) we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying first Abel's transformation and then using Lemma 2, we have that

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \\
\mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \right| + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \right| \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

Since

$$\mid T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \mid^k \leq 2^k (\mid T_{n,1}^{\alpha,\beta} \mid^k + \mid T_{n,2}^{\alpha,\beta} \mid^k),$$

in order to complete the proof of the Theorem, by (3) it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad \text{for} \quad r = 1, 2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$. We get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}\mid T_{n,1}^{\alpha,\beta}\mid^k &\leq \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha}A_v^{\beta}\theta_v^{\alpha,\beta}\Delta\lambda_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n-1}v^{\alpha k}v^{\beta k}\beta_v(\theta_v^{\alpha,\beta})^k\right\}\times\left\{\sum_{v=1}^{n-1}\beta_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\beta_v(\theta_v^{\alpha,\beta})^k\sum_{n=v+1}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}} \\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\beta_v(\theta_v^{\alpha,\beta})^k\int_v^{\infty}\frac{\mathrm{d}x}{x^{1+(\alpha+\beta)k}} \\
&= O(1)\sum_{v=1}^{m}\beta_v(\theta_v^{\alpha,\beta})^k = O(1)\sum_{v=1}^{m}v\beta_v\frac{1}{v}(\theta_v^{\alpha,\beta})^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{p=1}^{v}\frac{1}{p}(\theta_p^{\alpha,\beta})^k \\
&\quad +O(1)m\beta_m\sum_{v=1}^{m}\frac{1}{v}(\theta_v^{\alpha,\beta})^k \\
&= O(1)\sum_{v=1}^{m-1}v\mid\Delta\beta_v\mid X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m \\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

in view of hypotheses of the Theorem and Lemma 1.

Similarly we have that

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\mid\lambda_n\theta_n^{\alpha,\beta}\mid^k &= O(1)\sum_{n=1}^{m}\frac{\mid\lambda_n\mid}{n}(\theta_n^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta\mid\lambda_n\mid\sum_{v=1}^{n}\frac{1}{v}(\theta_v^{\alpha,\beta})^k \\
&\quad +O(1)\mid\lambda_m\mid\sum_{v=1}^{m}\frac{1}{v}(\theta_v^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\mid\Delta\lambda_n\mid X_n+O(1)\mid\lambda_m\mid X_m \\
&= O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)\mid\lambda_m\mid X_m \\
&= O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mid T_{n,r}^{\alpha,\beta}\mid^k<\infty\quad\text{for}\quad r=1,2.$$

This completes the proof of the Theorem. For $k=1$, the proof of the Theorem is obvious. If we take $\beta=0$, then we get a new result for $\mid C,\alpha\mid_k$ summability factors. Also, if we take $\beta=0$ and $\alpha=1$, then we have a result dealing with $\mid C,1\mid_k$ summability factors.

ACKNOWLEDGEMENT

The author wishes to express his sincerest thanks to the referees for their invaluable suggestions for the improvement of this paper.

REFERENCES

1. Aljancic, S. and Arandelovic, D. *O*-regularly varying functions. *Publ. Inst. Math.*, 1977, **22**, 5–22.
2. Bor, H. and Srivastava, H. M. Almost increasing sequences and their applications. *Int. J. Pure Appl. Math.*, 2002, **3**, 29–35.
3. Borwein, D. Theorems on some methods of summability. *Quart. J. Math., Oxford*, 1958, Ser. 9, 310–316.
4. Bosanquet, L. S. A mean value theorem. *J. London Math. Soc.*, 1941, **16**, 146–148.
5. Das, G. A Tauberian theorem for absolute summability. *Proc. Camb. Phil. Soc.*, 1970, **67**, 321–326.
6. Flett, T. M. On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc.*, 1957, **7**, 113–141.
7. Leindler, L. A new application of quasi power increasing sequences. *Publ. Math. (Debrecen)*, 2001, **58**, 791–796.
8. Pati, T. The summability factors of infinite series. *Duke Math. J.*, 1954, **21**, 271–284.

Astmeliselt kasvavate jadade uuest rakendusest

Hüseyin Bor

On tõestatud teoreem menetlusega $\mid C, \alpha, \beta \mid_k$ määratud summeeruvustegurite kohta varasemast nõrgematel eeldustel.