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MATHEMATICS

Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions

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Abstract. We study Riemannian manifolds M admitting a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is a parallel unit vector field with respect to the Levi-Civita connection ∇. We prove that $R\cdot\tilde{R}=0$ if and only if M is semisymmetric; if $\tilde{R}\cdot R=0$ or $R\cdot\tilde{R}-\tilde{R}\cdot R=0$ or M is semisymmetric and $\tilde{R}\cdot\tilde{R}=0$, then M is conformally flat and quasi-Einstein. Here R and $\tilde{R}$ denote the curvature tensors of ∇ and $\widetilde{\nabla}$, respectively.

Key words: Levi-Civita connection, semi-symmetric metric connection, conformally flat manifold, quasi-Einstein manifold.

1. INTRODUCTION

Let $\tilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T is given by

$$T(X,Y)=\tilde{\nabla}_XY-\tilde{\nabla}_YX-[X,Y].$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g in M such that $\tilde{\nabla}g=0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric [19]. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Hayden [9] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called a Hayden connection. In [8,13], Friedmann and Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a *semi-symmetric connection* if its torsion tensor T is of the form

$$T(X,Y)=\omega(Y)X-\omega(X)Y, \tag{1}$$

where the 1-form ω is defined by

$$\omega(X)=g(X,U),$$

and U is a vector field. In [12], Pak showed that a Hayden connection with the torsion tensor of the form (1) is a semi-symmetric metric connection. In [18], Yano considered a semi-symmetric metric connection and studied some of its properties. He proved that in order that a Riemannian manifold admits a semi-symmetric

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metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat. For some properties of Riemannian manifolds with a semi-symmetric metric connection see also [1,4,10,16,17].

In [14,15], Szabó studied semisymmetric Riemannian manifolds, that is Riemannian manifolds satisfying the condition $R \cdot R = 0$. It is well known that locally symmetric manifolds (i.e. Riemannian manifolds satisfying the condition $\nabla R = 0$) are trivially semisymmetric. But the converse statement is not true. According to Szabó, many geometrists have studied semisymmetric Riemannian manifolds.

Motivated by the studies of the above authors, in this paper we consider Riemannian manifolds (M,g) admitting a semi-symmetric metric connection such that U is a unit parallel vector field with respect to the Levi-Civita connection ∇. We investigate the conditions $R \cdot \tilde{R} = 0$, $\tilde{R} \cdot R = 0$, $R \cdot \tilde{R} - \tilde{R} \cdot R = 0$, and $\tilde{R} \cdot \tilde{R} = 0$ on M, where R and $\tilde{R}$ denote the curvature tensors of ∇ and $\widetilde{\nabla}$, respectively.

The paper is organized as follows. In Section 2 and Section 3, we give the necessary notions and results which will be used in the next sections. In Section 4, we prove that $R \cdot \tilde{R} = 0$ if and only if M is semisymmetric, if $\tilde{R} \cdot R = 0$ or $R \cdot \tilde{R} - \tilde{R} \cdot R = 0$ or M is semisymmetric and $\tilde{R} \cdot \tilde{R} = 0$, then M is conformally flat and quasi-Einstein.

2. PRELIMINARIES

An n-dimensional Riemannian manifold (M,g), $n > 2$, is said to be an *Einstein manifold* if its Ricci tensor S satisfies the condition $S = \frac{r}{n} g$, where r denotes the scalar curvature of M. If the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + bD(X)D(Y), \tag{2}$$

where a, b are scalars of which $b \neq 0$ and D is a non-zero 1-form, then M is called a *quasi-Einstein manifold* [3].

For a $(0,k)$-tensor field T, $k \geq 1$, on (M,g) we define the tensor $R \cdot T$ (see [5]) by

$$\begin{aligned}(R(X,Y) \cdot T)(X_1, ..., X_k) = & - T(R(X,Y)X_1, X_2, ..., X_k) \\ & - ... - T(X_1, ..., X_{k-1}, R(X,Y)X_k).\end{aligned} \tag{3}$$

If $R \cdot R = 0$, then M is called *semisymmetric* [14]. In addition, if E is a symmetric $(0,2)$-tensor field, then we define the $(0, k+2)$-tensor $Q(E,T)$ (see [5]) by

$$\begin{aligned}Q(E,T)(X_1, ..., X_k; X, Y) = & - T((X \wedge_E Y)X_1, X_2, ..., X_k) \\ & - ... - T(X_1, ..., X_{k-1}, (X \wedge_E Y)X_k),\end{aligned} \tag{4}$$

where $X \wedge_E Y$ is defined by

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y.$$

The *Weyl tensor* of a Riemannian manifold (M,g) is defined by

$$\begin{aligned}C(X,Y,Z,W) = & \; R(X,Y,Z,W) - \frac{1}{n-2}\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \right. \\ & \left. + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\right] \\ & + \frac{r}{(n-1)(n-2)}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right],\end{aligned}$$

where r denotes the scalar curvature of M. For $n \geq 4$, if $C = 0$, the manifold is called *conformally flat* [19].

Now we give the Lemmas which will be used in Section 4.

Lemma 2.1. [6] *Let* (M,g), $n \geq 3$, *be a semi-Riemannian manifold. Let at a point* $x \in M$ *be given a non-zero symmetric* $(0,2)$*-tensor* E *and a generalized curvature tensor* B *such that at* x *the following condition is satisfied*: $Q(E,B) = 0$. *Moreover, let* V *be a vector at* x *such that the scalar* $\rho = a(V)$ *is non-zero, where* a *is a covector defined by* $a(X) = E(X,V)$, $X \in T_xM$.
(i) *If* $E = \frac{1}{\rho} a \otimes a$, *then at* x *we have* $\sum_{X,Y,Z} a(X)B(Y,Z) = 0$, *where* $X,Y,Z \in T_xM$.
(ii) *If* $E - \frac{1}{\rho} a \otimes a$ *is non-zero, then at* x *we have* $B = \frac{\gamma}{2} E \wedge E$, $\gamma \in \mathbb{R}$. *Moreover, in both cases, at* x *we have* $B \cdot B = Q(Ric(B), B)$.

Lemma 2.2. [7] *Let* (M,g), $n \geq 4$, *be a semi-Riemannian manifold and* E *be the symmetric* $(0,2)$ *tensor at* $x \in M$ *defined by* $E = \alpha g + \beta \omega \otimes \omega$, $\omega \in T_x^*M$, $\alpha, \beta \in \mathbb{R}$. *If at* x *the curvature tensor* R *is expressed by* $R = \frac{\gamma}{2} E \wedge E$, $\gamma \in \mathbb{R}$, *then the Weyl tensor* C *vanishes at* x.

3. SEMI-SYMMETRIC METRIC CONNECTION

If ∇ is the Levi-Civita connection of a Riemannian manifold M, then we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X,Y)U,$$

where

$$\omega(X) = g(X,U),$$

and X,Y,U are vector fields on M. Let R and $\widetilde{R}$ denote the Riemannian curvature tensor of ∇ and $\widetilde{\nabla}$, respectively. Then we know that [18]

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) - \theta(Y,Z)g(X,W) \\ &\quad + \theta(X,Z)g(Y,W) - g(Y,Z)\theta(X,W) \\ &\quad + g(X,Z)\theta(Y,W), \end{aligned} \tag{5}$$

where

$$\theta(X,Y) = g(AX,Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2} g(X,Y).$$

Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection, i.e., $\nabla U = 0$ and $\|U\| = 1$. Then

$$(\nabla_X \omega)Y = \nabla_X \omega(Y) - \omega(\nabla_X Y) = 0. \tag{6}$$

So θ is a symmetric $(0,2)$-tensor field. Hence equation (5) can be written as

$$\widetilde{R} = R - g \barwedge \theta, \tag{7}$$

where $\barwedge$ is the Kulkarni–Nomizu product, which is defined by

$$\begin{aligned} (g \barwedge \theta)(X,Y,Z,W) &= g(Y,Z)\theta(X,W) - g(X,Z)\theta(Y,W) \\ &\quad + g(X,W)\theta(Y,Z) - g(Y,W)\theta(X,Z). \end{aligned}$$

Since U is a parallel unit vector field, it is easy to see that $\widetilde{R}$ is a generalized curvature tensor and it is trivial that $R(X,Y)U = 0$. Hence by a contraction we find $S(Y,U) = \omega(QY) = 0$, where S denotes the Ricci tensor of ∇ and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. It is easy to see that we also have the following relations:

$$\widetilde{\nabla}_X U = X - \omega(X)U,$$
$$\widetilde{R}(X,Y)U = 0, \quad \widetilde{R}\cdot\theta = 0, \tag{8}$$
$$\theta^2(X,Y) \overset{def}{:=} g(AX,AY) = \frac{1}{4}g(X,Y),$$

and

$$\tilde{S} = S - (n-2)(g - w\otimes w), \tag{9}$$
$$\widetilde{r} = r - (n-2)(n-1). \tag{10}$$

Using (7), (9), and (10), we get

$$\tilde{C} = C,$$

where $\tilde{S}$, $\tilde{C}$, and $\widetilde{r}$ denote the Ricci tensor, Weyl tensor, and the scalar curvature of M with respect to semi-symmetric connection $\tilde{\nabla}$.

4. MAIN RESULTS

The tensors $\tilde{R}\cdot R$ and $Q(\theta,T)$ are defined in the same way with (3) and (4). Let $(R\cdot\tilde{R})_{hijklm}$ and $(\tilde{R}\cdot R)_{hijklm}$ denote the local components of the tensors $R\cdot\tilde{R}$ and $\tilde{R}\cdot R$, respectively. Hence, we have the following proposition:

Proposition 4.1. *Let (M,g) be a Riemannian manifold admitting a semi-symmetric metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇, then*

$$(R\cdot\tilde{R})_{hijklm} = (R\cdot R)_{hijklm}, \tag{11}$$

$$(\tilde{R}\cdot R)_{hijklm} = (R\cdot R)_{hijklm} - Q(g - w\otimes w, R)_{hijklm}. \tag{12}$$

Proof. Since U is parallel, we have $R\cdot\theta = 0$. So from (7) we get

$$R\cdot\tilde{R} = R\cdot R - g\wedge R\cdot\theta = R\cdot R.$$

Applying (5) in (3) and using (4), we obtain

$$\begin{aligned}(\tilde{R}\cdot R)_{hijklm} &= (R\cdot R)_{hijklm} - Q(\theta,R)_{hijklm}\\ &\quad - \frac{1}{2}(g_{hl}R_{mijk} - g_{hm}R_{lijk} - g_{il}R_{mhjk}\\ &\quad + g_{im}R_{lhjk} + g_{jl}R_{mkhi} - g_{jm}R_{lkhi}\\ &\quad - g_{kl}R_{mjhi} + g_{km}R_{ljhi})\\ &= (R\cdot R)_{hijklm} - Q\Big(\theta + \frac{1}{2}g, R\Big)_{hijklm}\\ &= (R\cdot R)_{hijklm} - Q(g - w\otimes w, R)_{hijklm}.\end{aligned} \tag{13}$$

This completes the proof of the proposition. □

As an immediate consequence of Proposition 4.1 we have the following theorem:

Theorem 4.2. *Let (M,g) be a Riemannian manifold admitting a semi-symmetric metric connection and U be a parallel unit vector field with respect to the Levi-Civita connection ∇. Then $R\cdot\tilde{R}=0$ if and only if M is semisymmetric.*

Theorem 4.3. *Let (M,g) be a semisymmetric $n>3$ dimensional Riemannian manifold admitting a semi-symmetric metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $\tilde{R}\cdot R=0$, then M is a conformally flat quasi-Einstein manifold.*

Proof. Since the condition $\tilde{R}\cdot R=0$ holds on M, from Proposition 4.1 we have

$$Q(g-\omega\otimes\omega,R)_{hijklm}=0. \tag{14}$$

So we have two possibilities:

$$\operatorname{rank}(g-\omega\otimes\omega)=1 \tag{15}$$

or

$$\operatorname{rank}(g-\omega\otimes\omega)>1. \tag{16}$$

Suppose that (15) holds at a point x. Thus we have

$$g-\omega\otimes\omega=\rho z\otimes z,$$

where $z\in T_x^*M$ and $\rho\in\mathbb{R}$. Because of non-zero coefficient of g, this relation does not occur. Thus the case (16) must be fulfilled at x. By virtue of Lemma 2.1, (14) gives us

$$R=\frac{\gamma}{2}((g-\omega\otimes\omega)\wedge(g-\omega\otimes\omega)),\quad \gamma\neq 0,\quad \gamma\in\mathbb{R}.$$

So from Lemma 2.2 we obtain $C=0$, which gives us that M is conformally flat. Moreover, contracting (14) with g^{ij}, we get

$$Q(g-\omega\otimes\omega,S)_{hklm}=0,$$

which gives us

$$S=\lambda(g-\omega\otimes\omega),$$

where $\lambda=\frac{r}{n-1}:M\to\mathbb{R}$ is a function. So by virtue of (2), M is quasi-Einstein. Thus the proof of the theorem is completed. □

Theorem 4.4. *Let (M,g) be a Riemannian manifold admitting a semi-symmetric metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $R\cdot\tilde{R}-\tilde{R}\cdot R=0$, then M is a conformally flat quasi-Einstein manifold.*

Proof. Using (11) and (12), we get

$$Q(g-\omega\otimes\omega,R)_{hijklm}=0.$$

Using the same method in the proof of Theorem 4.3, we obtain that M is conformally flat and quasi-Einstein. So we get the result as required. □

Theorem 4.5. *Let (M,g) be an $n>3$ dimensional semisymmetric Riemannian manifold admitting a semi-symmetric metric connection. If U is a parallel vector field with respect to the Levi-Civita connection ∇ and $\tilde{R}\cdot\tilde{R}=0$, then M is conformally flat and quasi-Einstein.*

Proof. From (5) we have

$$\begin{aligned}(\tilde{R}\cdot\tilde{R})_{hijklm} &= (\tilde{R}\cdot R)_{hijklm}-(g\wedge\tilde{R}\theta)_{hijklm}\\ &= (R\cdot R)_{hijklm}-Q(g-w\otimes w,R)_{hijklm}\\ &\quad -g_{hk}(\tilde{R}\cdot\theta)_{ijlm}-g_{ij}(\tilde{R}\cdot\theta)_{hklm}\\ &\quad +g_{hj}(\tilde{R}\cdot\theta)_{iklm}+g_{ik}(\tilde{R}\cdot\theta)_{hjlm}.\end{aligned} \tag{17}$$

Using (8), equation (17) is reduced to

$$(\tilde{R}\cdot\tilde{R})_{hijklm} = (R\cdot R)_{hijklm} - Q(g-\omega\otimes\omega,R)_{hijklm}.$$

We suppose that $(\tilde{R}\cdot\tilde{R})_{hijklm} = 0$ and M is semisymmetric. Using the same method in the proof of Theorem 4.3, we obtain that M is conformally flat and quasi-Einstein. This proves the theorem. □

The following example shows that there is a Riemannian manifold with a semi-symmetric metric connection having a parallel vector field associated to the 1-form satisfying $R\cdot\widetilde{R} = R\cdot R$.

Example. Let M^{2m+1} be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ,ξ,η), that is, φ is a (1,1)-tensor field, ξ is a vector field, and η is a 1-form such that

$$\varphi^2 = I - \eta\otimes\xi \quad \text{and} \quad \eta(\xi) = 1.$$

Then

$$\varphi(\xi) = 0 \quad \text{and} \quad \eta\circ\varphi = 0.$$

Let g be a compatible Riemannian metric with (φ,ξ,η), that is,

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

or, equivalently,

$$g(X,\varphi Y) = -g(\varphi X;Y) \quad \text{and} \quad g(X,\xi) = \eta(X)$$

for all $X,Y\in\chi(M)$. Then M^{2m+1} becomes an almost contact metric manifold equipped with an almost contact metric structure (φ,ξ,η,g). An almost contact metric manifold is *cosymplectic* [2] if $\nabla_X\varphi = 0$. From the formula $\nabla_X\varphi = 0$ it follows that

$$\nabla_X\xi = 0, \quad \nabla_X\eta = 0, \quad \text{and} \quad R(X,Y)\xi = 0.$$

So we have the following relations:

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$
$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi,$$
$$\theta = \frac{1}{2}g - \eta\otimes\eta.$$

Hence $\nabla\theta = 0$ and $R\cdot\theta = 0$, which gives us $R\cdot\widetilde{R} = R\cdot R$.

A cosymplectic manifold M is said to be a *cosymplectic space form* if the φ-sectional curvature is constant c along M. A cosymplectic space form will be denoted by $M(c)$. Then the Riemannian curvature tensor R on $M(c)$ is given by [11]

$$\begin{aligned} 4R(X,Y,Z,W) &= c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &\quad + g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) \\ &\quad - 2g(X,\varphi Y)g(Z,\varphi W) - g(X,W)\eta(Y)\eta(Z) \\ &\quad + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &\quad + g(Y,W)\eta(X)\eta(Z)\}. \end{aligned}$$

From direct calculation we get

$$S(X,W) = \frac{nc}{2}\{g(X,W) - \eta(X)\eta(W)\},$$

which gives us that M is quasi-Einstein.

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Teatavaid semisümmeetrilisuse tingimusi rahuldavad Riemanni muutkonnad semi-sümmeetrilise meetrilise seostusega

Cengizhan Murathan ja Cihan Özgür

On uuritud Riemanni muutkondi M sellise semi-sümmeetrilise meetrilise seostusega $\widetilde{\nabla}$, et vektorväli U on ühikvektorväli, mis on paralleelne Levi-Civita seostuse ∇ suhtes. On tõestatud, et $R\cdot\tilde{R} = 0$ siis ja ainult siis, kui M on semisümmeetriline; kui $\tilde{R}\cdot R = 0$ või $R\cdot\tilde{R} - \tilde{R}\cdot R = 0$ või M on semisümmeetriline ja $\tilde{R}\cdot\tilde{R} = 0$, siis M on konformselt tasane ja kvaasi-Einstein; siin R ja $\tilde{R}$ on vastavalt ∇ ning $\widetilde{\nabla}$ kõverustensorid.