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MATHEMATICS

Some characterizations of Mannheim partner curves in the Minkowski 3-space E_1^3

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Abstract. In this paper, we give some characterizations of Mannheim partner curves in the Minkowski 3-space E_1^3. Firstly, we classify these curves in E_1^3. Next, we give some relationships characterizing these curves and we show that the Mannheim theorem is not valid for the Mannheim partner curves in E_1^3. Moreover, by considering the spherical indicatrix of the Frenet vectors of those curves, we obtain some new relationships between the curvatures and torsions of the Mannheim partner curves in E_1^3.

Key words: Minkowski 3-space, timelike curve, spacelike curve, Mannheim partner curves.

1. INTRODUCTION

In differential geometry, special curves have an important role. Especially the partner curves, i.e., the curves which are related to each other at the corresponding points, have attracted the attention of many mathematicians. Well-known partner curves are the Bertrand curves, which are defined by the property that at the corresponding points of two space curves the principal normal vectors are common. Bertrand partner curves are studied in refs [1–4,13,15]. Ravani and Ku transported the notion of Bertrand curves to the ruled surfaces and called them Bertrand offsets [12]. Recently, Liu and Wang [5,14] defined a new curve pair for space curves. They called these new curves Mannheim partner curves: Let x and x_1 be two curves in the three-dimensional Euclidean space E^3. If there exists a correspondence between the space curves x and x_1 such that, at the corresponding points of the curves, the principal normal lines of x coincide with the binormal lines of x_1, then x is called a Mannheim curve, and x_1 is called a Mannheim partner curve of x. The pair $\{x, x_1\}$ is said to be a Mannheim pair. They showed that the curve $x_1(s_1)$ is the Mannheim partner curve of the curve $x(s)$ if and only if the curvature κ_1 and the torsion τ_1 of $x_1(s_1)$ satisfy the following equation

$$\dot{\tau} = \frac{d\tau}{ds_1} = \frac{\kappa_1}{\lambda}(1 + \lambda^2 \tau_1^2)$$

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for some non-zero constants λ. They also studied the Mannheim partner curves in the Minkowski 3-space and obtained the necessary and sufficient conditions for the Mannheim partner curves in E_1^3 (see [5] and [14] for details). Moreover, Oztekin and Ergüt [11] studied the null Mannheim curves in the same space. Orbay and Kasap [10] gave new characterizations of Mannheim partner curves in Euclidean 3-space. They also studied [9] the Mannheim offsets of ruled surfaces in Euclidean 3-space. The corresponding characterizations of Mannheim offsets of timelike and spacelike ruled surfaces were given by Onder et al. [6,7].

In this paper, we give new characterizations of Mannheim partner curves in the Minkowski 3-space E_1^3. Next, we show that the Mannheim theorem is not valid for the Mannheim partner curves in E_1^3. Moreover, we give some new characterizations of the Mannheim partner curves by considering the spherical indicatrix of some Frenet vectors of the curves.

2. PRELIMINARIES

The Minkowski 3-space E_1^3 is the real vector space E^3 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. According to this metric, in E_1^3 an arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$, and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$ [8]. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike, or null (lightlike) if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike, or null (lightlike), respectively. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For the vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \wedge \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2,\ x_1 y_3 - x_3 y_1,\ x_2 y_1 - x_1 y_2),$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \qquad e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \text{ and } e_1 \wedge e_2 = -e_3,\ e_2 \wedge e_3 = e_1,\ e_3 \wedge e_1 = -e_2.$$

The Lorentzian sphere and hyperbolic sphere of radius r and centre 0 in E_1^3 are given by

$$S_1^2 = \{\vec{x} = (x_1, x_2, x_3) \in E_1^3 : \langle \vec{x}, \vec{x} \rangle = r^2\}$$

and

$$H_0^2 = \{\vec{x} = (x_1, x_2, x_3) \in E_1^3 : \langle \vec{x}, \vec{x} \rangle = -r^2\},$$

respectively [6,7].

Denote by $\{\vec{T}, \vec{N}, \vec{B}\}$ the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski space E_1^3. For an arbitrary spacelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}, \tag{1a}$$

where $g(\vec{T}, \vec{T}) = 1$, $g(\vec{N}, \vec{N}) = \varepsilon = \pm 1$, $g(\vec{B}, \vec{B}) = -\varepsilon$, $g(\vec{T}, \vec{N}) = g(\vec{T}, \vec{B}) = g(\vec{N}, \vec{B}) = 0$, and k_1 and k_2 are curvature and torsion of the spacelike curve $\alpha(s)$, respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$. Furthermore, for a timelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}, \tag{1b}$$

where $g(\vec{T},\vec{T}) = -1$, $g(\vec{N},\vec{N}) = g(\vec{B},\vec{B}) = 1$, $g(\vec{T},\vec{N}) = g(\vec{T},\vec{B}) = g(\vec{N},\vec{B}) = 0$, and k_1 and k_2 are curvature and torsion of the timelike curve $\alpha(s)$, respectively (see [8] and [16] for details).

Definition 2.1.

(i) **Hyperbolic angle:** *Let $\vec{x}$ and $\vec{y}$ be future pointing (or past pointing) timelike vectors in E_1^3. Then there is a unique real number $\theta \geq 0$ such that $<\vec{x},\vec{y}> = -|\vec{x}||\vec{y}|\cosh\theta$. This number is called the hyperbolic angle between the vectors $\vec{x}$ and $\vec{y}$.*

(ii) **Central angle:** *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in E_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $<\vec{x},\vec{y}> = |\vec{x}||\vec{y}|\cosh\theta$. This number is called the central angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iii) **Spacelike angle:** *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in E_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $<\vec{x},\vec{y}> = |\vec{x}||\vec{y}|\cos\theta$. This number is called the spacelike angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iv) **Lorentzian timelike angle:** *Let $\vec{x}$ be a spacelike vector and $\vec{y}$ be a timelike vector in E_1^3. Then there is a unique real number $\theta \geq 0$ such that $<\vec{x},\vec{y}> = |\vec{x}||\vec{y}|\sinh\theta$. This number is called the Lorentzian timelike angle between the vectors $\vec{x}$ and $\vec{y}$* [6,7].

In this paper, we study the Mannheim partner curves in E_1^3. We obtain the relationships between the curvatures and torsions of the Mannheim partner curves with respect to each other. Using these relationships, we give the Mannheim theorem for the Mannheim partner curves in the Minkowski 3-space E_1^3.

3. MANNHEIM PARTNER CURVES IN THE MINKOWSKI 3-SPACE E_1^3

In this section, by considering the Frenet frames, we give the characterizations of the Mannheim partner curves in the Minkowski 3-space E_1^3.

Definition 3.1. *Let C and C^* be two curves in the Minkowski 3-space E_1^3 given by the parametrizations $\alpha(s)$ and $\alpha^*(s^*)$, respectively, and let them have at least four continuous derivatives. If there exists a correspondence between the space curves C and C^* such that the principal normal lines of C coincide with the binormal lines of C^* at the corresponding points of curves, then C is called a Mannheim curve and C^* is called a Mannheim partner curve of C. The pair $\{C, C^*\}$ is said to be a Mannheim pair* [5].

By considering the Lorentzian casual characters of the curves, it is easily seen from Definition 3.1 that there are five different types of the Mannheim partner curves in the Minkowski 3-space E_1^3. Let the pair $\{C, C^*\}$ be a Mannheim pair. Then according to the characters of the curves C and C^* we have the following cases:

Case 1. The curve C^* is timelike.

If the curve C^* is timelike, then there are two cases.

(i) The curve C is a spacelike curve with a timelike principal normal. In this case, we say that the pair $\{C, C^*\}$ is a Mannheim pair of type 1.

(ii) The curve C is a timelike curve. In this case, we say that the pair $\{C, C^*\}$ is a Mannheim pair of type 2.

Case 2. The curve C^* is spacelike.
If the curve C^* is a spacelike curve, then there are three cases.
(iii) The curve C^* is a spacelike curve with a timelike binormal vector and the curve C is a spacelike curve with a timelike principal normal vector. In this case, we say that the pair $\{C, C^*\}$ is a Mannheim pair of type 3.
(iv) The curve C^* is a spacelike curve with a timelike binormal vector and the curve C is a timelike curve. In this case, we say that the pair $\{C, C^*\}$ is a Mannheim pair of type 4.
(v) The curve C^* is a spacelike curve with a timelike principal normal vector and the curve C is a spacelike curve with a timelike binormal vector. In this case, we say that the pair $\{C, C^*\}$ is a Mannheim pair of type 5.

Theorem 3.1. *The distance between the corresponding points of the Mannheim partner curves is constant in* E_1^3.

Proof. Let us consider the case when the pair $\{C, C^*\}$ is a Mannheim pair of type 1. From Definition 3.1 we can write

$$\vec{\alpha}(s) = \vec{\alpha}^*(s^*) + \lambda(s^*)\vec{B}^*(s^*) \tag{2}$$

for some function $\lambda(s^*)$. By taking the derivative of Equation (2) with respect to s^* and using Equations (1), we obtain

$$\vec{T}\frac{ds}{ds^*} = \vec{T}^* + \lambda\tau^*\vec{N}^* + \dot{\lambda}\vec{B}^*. \tag{3}$$

Since $\vec{N}$ and $\vec{B}^*$ are linearly dependent, we have $\left\langle \vec{T}, \vec{B}^* \right\rangle = 0.$ Then, we get

$$\dot{\lambda} = 0.$$

This means that λ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$d(\alpha^*(s^*), \alpha(s)) = \left\|\alpha(s) - \alpha^*(s^*)\right\| = \left\|\lambda\vec{B}^*\right\| = |\lambda|.$$

Namely, $d(\alpha^*(s^*), \alpha(s)) = \text{constant}.$ For the other cases, we obtain the same result. □

Theorem 3.2. *For a curve* C *in* E_1^3, *there is a curve* C^* *such that* $\{C, C^*\}$ *is a Mannheim pair.*

Proof. Since $\vec{N}$ and $\vec{B}^*$ are linearly dependent for all types, Equation (2) can be written as

$$\vec{\alpha}^* = \vec{\alpha} - \lambda\vec{N}. \tag{4}$$

Now, there is a curve C^* for all values of nonzero constant λ. □

Theorem 3.3. *Let* $\{C, C^*\}$ *be a Mannheim pair in* E_1^3. *Then the relations between the curvatures and torsions of the curves* C, C^* *are given as follows*:
(i) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 1 *or* 4, *then*

$$\tau^* = \frac{-\kappa}{\lambda\tau}.$$

(ii) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 2, 3, *or* 5, *then*

$$\tau^* = \frac{\kappa}{\lambda\tau}.$$

Proof. (i) Let the pair $\{C, C^*\}$ be a Mannheim pair of type 1. By considering the nonzero constant λ in Equation (3), we obtain

$$\vec{T}\frac{ds}{ds^*} = \vec{T}^* + \lambda\tau^*\vec{N}^*. \tag{5}$$

Considering Definition 2.1, we have

$$\begin{cases} \vec{T} = \sinh\theta\vec{T}^* + \cosh\theta\vec{N}^* \\ \vec{B} = \cosh\theta\vec{T}^* + \sinh\theta\vec{N}^* \end{cases}, \tag{6}$$

where θ is the angle between the tangent vectors $\vec{T}$ and $\vec{T}^*$ at the corresponding points of the curves C and C^*. From Equations (5) and (6), we get

$$\cosh\theta = \lambda\tau^*\frac{ds^*}{ds}, \quad \sinh\theta = \frac{ds^*}{ds}. \tag{7}$$

By considering Equation (1), the derivative of Equation (4) with respect to s^* gives us the following

$$\vec{T}^* = (1-\lambda\kappa)\frac{ds}{ds^*}\vec{T} - \lambda\tau\frac{ds}{ds^*}\vec{B}. \tag{8}$$

From Equation (6), we get

$$\begin{cases} \vec{T}^* = -\sinh\theta\vec{T} + \cosh\theta\vec{B} \\ \vec{N}^* = \cosh\theta\vec{T} - \sinh\theta\vec{B} \end{cases}. \tag{9}$$

From Equations (8) and (9), we obtain

$$\cosh\theta = -\lambda\tau\frac{ds}{ds^*}, \qquad \sinh\theta = (\lambda\kappa - 1)\frac{ds}{ds^*}. \tag{10}$$

Then by Equations (7) and (10), we see that

$$\cosh^2\theta = -\lambda^2\tau\tau^*, \qquad \sinh^2\theta = \lambda\kappa - 1,$$

which gives us

$$\tau^* = \frac{-\kappa}{\lambda\tau}.$$

The proof of the statement given in (ii) can be given in a similar way. □

Theorem 3.4. *Let $\{C, C^*\}$ be a Mannheim pair in E_1^3. The relationship between the curvature and torsion of the curve C is given as follows*:
(i) *If the pair $\{C, C^*\}$ is a Mannheim pair of type* 1, 2, *or* 5, *then we have*

$$\mu\tau + \lambda\kappa = 1.$$

(ii) *If the pair $\{C, C^*\}$ is a Mannheim pair of type* 3 *or* 4, *then the relationship is given by*

$$\mu\tau - \lambda\kappa = 1,$$

where λ and μ are nonzero real numbers.

Proof. (i) Assume that the pair $\{C, C^*\}$ is a Mannheim pair of type 1. Then, from Equation (10), we have

$$\frac{-\cosh\theta}{\lambda\tau} = \frac{-\sinh\theta}{1-\lambda\kappa},$$

and so, we get

$$1-\lambda\kappa = \lambda(\tanh\theta)\tau,$$

which gives us

$$\mu\tau + \lambda\kappa = 1,$$

where λ and $\mu = \lambda\tanh\theta$ are nonzero constants.

The proof of statement (ii) can be given in the same way. □

Theorem 3.5. *Let* $\{C, C^*\}$ *be a Mannheim pair in* E_1^3. *Then, the relationships between the curvatures and the torsions of the curves* C *and* C^* *are given as follows*:

(a) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 1, *then*

$$\text{(i) } \kappa^* = -\frac{d\theta}{ds^*}, \quad \text{(ii) } \tau^* = \kappa\cosh\theta + \tau\sinh\theta, \quad \text{(iii) } \kappa = \tau^*\cosh\theta, \quad \text{(iv) } \tau = -\tau^*\sinh\theta.$$

(b) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 2, *then*

$$\text{(i) } \kappa^* = -\frac{d\theta}{ds^*}, \quad \text{(ii) } \tau^* = -\kappa\sinh\theta - \tau\cosh\theta, \quad \text{(iii) } \kappa = \tau^*\sinh\theta, \quad \text{(iv) } \tau = -\tau^*\cosh\theta.$$

(c) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 3, *then*

$$\text{(i) } \kappa^* = -\frac{d\theta}{ds^*}, \quad \text{(ii) } \tau^* = -\kappa\sinh\theta + \tau\cosh\theta, \quad \text{(iii) } \kappa = \tau^*\sinh\theta, \quad \text{(iv) } \tau = \tau^*\cosh\theta.$$

(d) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 4, *then*

$$\text{(i) } \kappa^* = \frac{d\theta}{ds^*}, \quad \text{(ii) } \tau^* = \kappa\cosh\theta - \tau\sinh\theta, \quad \text{(iii) } \kappa = \tau^*\cosh\theta, \quad \text{(iv) } \tau = \tau^*\sinh\theta.$$

(e) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 5, *then*

$$\text{(i) } \kappa^* = -\frac{d\theta}{ds^*}, \quad \text{(ii) } \tau^* = \kappa\sin\theta + \tau\cos\theta, \quad \text{(iii) } \kappa = \tau^*\sin\theta, \quad \text{(iv) } \tau = \tau^*\cos\theta.$$

Proof. (a) Let the pair $\{C, C^*\}$ be a Mannheim pair of type 1 in the Minkowski 3-space.

(i) By taking the derivative of the equation $\left\langle \vec{T}, \vec{T}^* \right\rangle = \sinh\theta$ with respect to s^*, we have

$$\left\langle \kappa\vec{N}, \vec{T}^* \right\rangle + \left\langle \vec{T}, \kappa^*\vec{N}^* \right\rangle = \cosh\theta\frac{d\theta}{ds^*}.$$

Furthermore, by considering $\vec{N}$ and $\vec{B}^*$ as linearly dependent and using Equations (2) and (9), we have

$$\kappa^* = -\frac{d\theta}{ds^*}.$$

By considering the equations $\left\langle \vec{N}, \vec{N}^* \right\rangle = 0$, $\left\langle \vec{T}, \vec{B}^* \right\rangle = 0$, and $\left\langle \vec{B}, \vec{B}^* \right\rangle = 0$, the proofs of the statements (ii), (iii), and (iv) of (a) in Theorem 3.5 can be given in a similar way of the proof of statement (i). □

From statements (iii) and (iv) of Theorem 3.5, we obtain the following result.

Proposition 3.1. *The torsion of the curve* C^* *is given by*

$$\tau^* = \kappa^2 - \tau^2.$$

The statements (b), (c), (d), and (e) can be proved as given in the proof of the statement (a).

Theorem 3.6. *Let* $\{C, C^*\}$ *be a Mannheim pair in* E_1^3. *For the corresponding points* $\alpha(s)$ *and* $\alpha^*(s^*)$ *of the curves* C, C^* *and for the curvature centres* M *and* M^* *at these points, the ratio*

$$\frac{\left\|\alpha^*(s^*)M\right\|}{\left\|\alpha(s)M\right\|} : \frac{\left\|\alpha^*(s^*)M^*\right\|}{\left\|\alpha(s)M^*\right\|}$$

is not constant.

Proof. Assume that the pair $\{C, C^*\}$ is a Mannheim pair of type 1. Then, we obtain the following equations:

$$\left\|\alpha(s)M\right\| = \frac{1}{\kappa}, \quad \left\|\alpha^*(s^*)M^*\right\| = \frac{1}{\kappa^*}, \quad \left\|\alpha(s)M^*\right\| = \sqrt{\left|\lambda^2 - \frac{1}{(\kappa^*)^2}\right|}, \quad \left\|\alpha^*(s^*)M\right\| = \frac{1}{\kappa} - \lambda.$$

So, we have

$$\frac{\left\|\alpha^*(s^*)M\right\|}{\left\|\alpha(s)M\right\|} : \frac{\left\|\alpha^*(s^*)M^*\right\|}{\left\|\alpha(s)M^*\right\|} = (1 - \lambda\kappa)\sqrt{\left|\lambda^2(\kappa^*)^2 - 1\right|} \neq \text{constant}.$$

If the pair $\{C, C^*\}$ is a Mannheim pair of type 2, 3, 4, or 5, we again find that the ratio is not constant. □

Proposition 3.2. *The Mannheim theorem is invalid for the Mannheim curves in* E_1^3.

Theorem 3.7. *Let the spherical indicatrix of the principal normal vector of the curve* C *be denoted by* C_2 *with the arclength parameter* s_2 *and let the spherical indicatrix of the binormal vector of the curve* C^* *be denoted by* C_3^* *with the arclength parameter* s_3^*. *If* $\{C, C^*\}$ *is a Mannheim pair in* E_1^3, *then we have the following*:

(i) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 1, *we have*

$$\kappa\frac{ds}{ds_2} = \tau^*\frac{ds^*}{ds_3^*}\cosh\theta, \quad \tau\frac{ds}{ds_2} = -\tau^*\frac{ds^*}{ds_3^*}\sinh\theta.$$

(ii) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 2 *or* 3, *we get*

$$\kappa\frac{ds}{ds_2} = -\tau^*\frac{ds^*}{ds_3^*}\sinh\theta, \quad \tau\frac{ds}{ds_2} = -\tau^*\frac{ds^*}{ds_3^*}\cosh\theta.$$

(iii) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 4, we have

$$\kappa\frac{ds}{ds_2} = -\tau^*\frac{ds^*}{ds_3^*}\cosh\theta, \quad \tau\frac{ds}{ds_2} = \tau^*\frac{ds^*}{ds_3^*}\sinh\theta.$$

(iv) *If the pair* $\{C, C^*\}$ *is a Mannheim pair of type* 5, *we have*

$$\kappa\frac{ds}{ds_2}=\tau^*\frac{ds^*}{ds_3^*}\sin\theta,\quad \tau\frac{ds}{ds_2}=\tau^*\frac{ds^*}{ds_3^*}\cos\theta.$$

Proof. (i) Suppose that the pair $\{C, C^*\}$ is a Mannheim pair of type 1. Let $\vec{T}_2$ be the tangent vector of the spherical indicatrix of the principal normal vector of the curve C and let $\vec{T}_3^*$ be the tangent vector of the spherical indicatrix of the binormal vector of the curve C^*. Since $\vec{N}$ and $\vec{B}^*$ are linearly dependent, the spherical indicatrix of the principal normal of the curve C is the same with the spherical indicatrix of the binormal of the curve C^*. Subsequently, we have

$$\vec{T}_2=\vec{N}'=(\kappa\vec{T}+\tau\vec{B})\frac{ds}{ds_2}$$

and

$$\vec{T}_3^*=\vec{B}^{*\prime}=\tau^*\vec{N}^*\frac{ds^*}{ds_3^*}.$$

Since $\vec{N}$ and $\vec{B}^*$ are linearly dependent, we can assume that

$$\vec{T}_2=\vec{T}_3^*.$$

Thus, we obtain the following equations:

$$\kappa\sinh\theta=-\tau\cosh\theta,\quad \kappa\frac{ds}{ds_2}\cosh\theta+\tau\frac{ds}{ds_2}\sinh\theta=\tau^*\frac{ds^*}{ds_3^*},$$

which gives us

$$\kappa\frac{ds}{ds_2}=\tau^*\frac{ds^*}{ds_3^*}\cosh\theta,\quad \tau\frac{ds}{ds_2}=-\tau^*\frac{ds^*}{ds_3^*}\sinh\theta,$$

which are desired equalities.

The proofs of the statements (ii), (iii), and (iv) of Theorem 3.7 can be given in a similar way. □

Example 1. Let us consider the spacelike curve (C^*) given by the parametrization

$$\alpha^*(s)=\left(-\frac{1}{2}\sinh s,\frac{1}{2}\cosh s,\frac{\sqrt{5}}{2}s\right).$$

The Frenet vectors of $\alpha^*(s)$ are obtained as follows:

$$\vec{T}^*=\left(-\frac{1}{2}\cosh s,\frac{1}{2}\sinh s,\frac{\sqrt{5}}{2}\right),$$
$$\vec{N}^*=(-\sinh s,\cosh s,0),$$
$$\vec{B}^*=\left(-\frac{\sqrt{5}}{2}\cosh s,\frac{\sqrt{5}}{2}\sinh s,\frac{1}{2}\right).$$

For $\lambda=20$, the parametric representation of the Mannheim partner curve (C) of the curve $\alpha^*(s)$ is obtained as

$$\alpha = \left(-\frac{1}{2}\sinh s - 10\sqrt{5}\cosh s, \frac{1}{2}\cosh s + 10\sqrt{5}\sinh s, \frac{\sqrt{5}}{2}s + 10 \right).$$

Then, the pair $\{C, C^*\}$ is a Mannheim pair of type 3. Figure 1 shows the different appearances of the curves α^* and α in space.

Example 2. Let us now consider the timelike curve (C^*) given by the parametrization

$$\alpha^*(s) = (2\sinh s,\, 2\cosh s,\, \sqrt{3}s).$$

The Frenet vectors of $\alpha^*(s)$ are obtained as follows:

$$\begin{aligned}
\vec{T}^* &= (2\cosh s, 2\sinh s, \sqrt{3}),\\
\vec{N}^* &= (\sinh s, \cosh s, 0),\\
\vec{B}^* &= (-\sqrt{3}\cosh s, -\sqrt{3}\sinh s, -2).
\end{aligned}$$

Then for $\lambda = 20,$ the Mannheim partner curve (C) of the curve $\alpha^*(s)$ is obtained as

$$\alpha = (2\sinh s - 20\sqrt{3}\cosh s, 2\cosh s - 20\sqrt{3}\sinh s, \sqrt{3}s - 40).$$

Then, the pair $\{C, C^*\}$ is a Mannheim pair of type 1. Figure 2 shows the different appearances of the curves α^* and α in space.

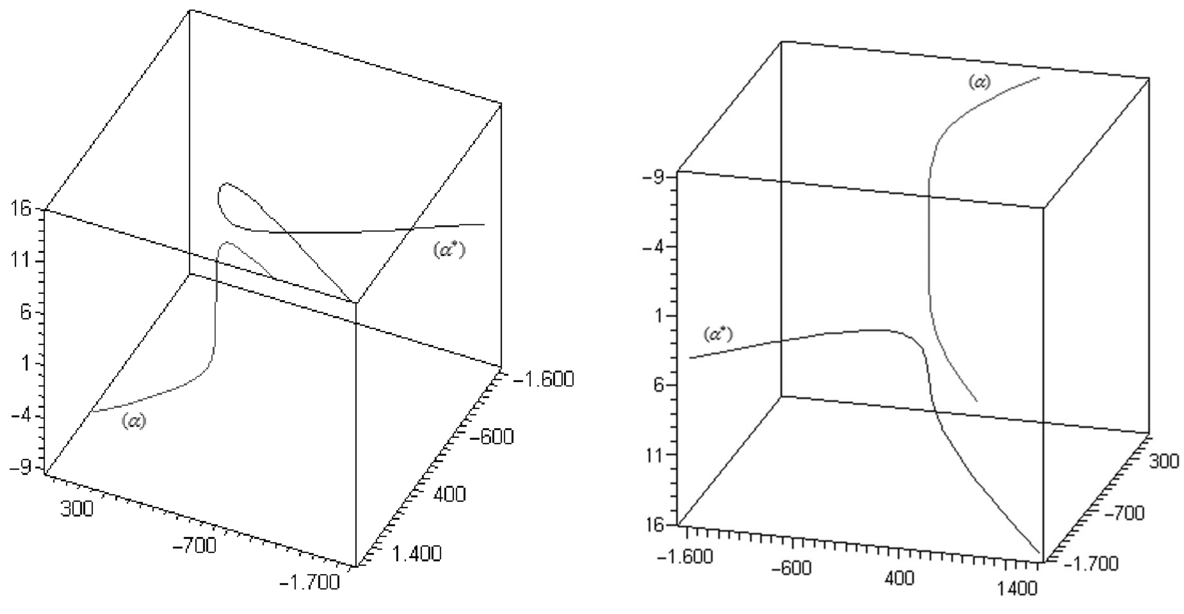

Fig. 1. The spacelike curve α* and its Mannheim partner curve α.

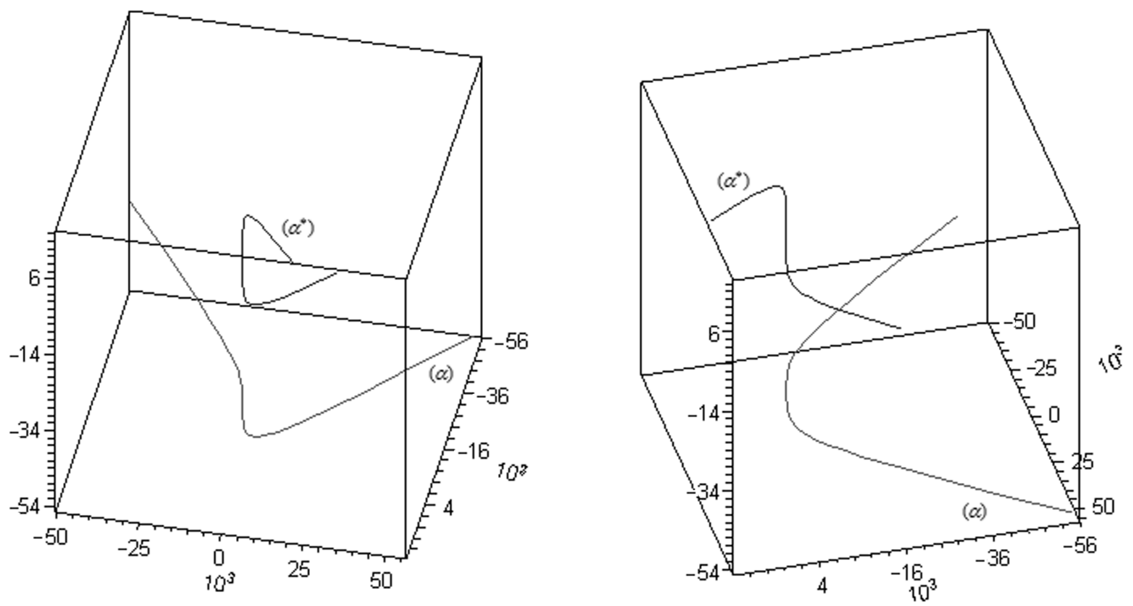

Fig. 2. The timelike curve α^* and its Mannheim partner curve α.

4. CONCLUSIONS

In this paper, we give some characterizations of the Mannheim partner curves in the Minkowski 3-space E_1^3. Moreover, we show that the Mannheim theorem is not valid for the Mannheim partner curves in E_1^3. Also, by considering the spherical indicatrix of some Frenet vectors of the Mannheim curves we give some new characterizations for these curves.

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Mannheimi partnerkõverate iseloomustus Minkowski 3-ruumis E_1^3

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Käesolevas töös anname Mannheimi partnerkõverate iseloomustuse Minkowski 3-ruumis. Kõigepealt me klassifitseerime need kõverad ruumis. Seejärel esitame mõned seosed, mis iseloomustavad neid kõveraid, ja näitame, et Mannheimi teoreem ei kehti Mannheimi partnerkõverate jaoks ruumis. Veelgi enam, uurides nende kõverate Frenet' vektorite sfäärilist indikatrissi, saame uusi seoseid Mannheimi partnerkõverate kõveruste ja väänete vahel.