Proceedings of the Estonian Academy of Sciences,
2011, **60**, 4, 221–237
doi: 10.3176/proc.2011.4.03
Available online at www.eap.ee/proceedings

MATHEMATICS

# Morita theorems for partially ordered monoids

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Received 23 December 2010, revised 9 February 2011, accepted 10 February 2011

**Abstract.** Two partially ordered monoids $S$ and $T$ are called Morita equivalent if the categories of right $S$-posets and right $T$-posets are Pos-equivalent as categories enriched over the category Pos of posets. We give a description of Pos-prodense biposets and prove Morita theorems I, II, and III for partially ordered monoids.

**Key words:** pomonoid, Morita equivalence, $S$-poset, Morita context.

## 1. INTRODUCTION

At the beginning of the 1970s, Knauer [5] and Banaschewski [2] proved the first fundamental results about Morita equivalence of monoids, establishing a theory parallel to the classical theory of Morita equivalent rings (see [1] for an overview about that). An overview of Morita theory of monoids can be found in [4]. The aim of this paper is to develop a theory of Morita equivalent partially ordered monoids (shortly pomonoids). In particular, we prove the analogues of theorems, which (at least in the ring case, see [7]) are usually called Morita I, Morita II, and Morita III. In Morita I we show that the endomorphism pomonoid $S$ of a cyclic projective generator over a pomonoid $T$ is Morita equivalent to $T$. In Morita II we prove that the functors that induce a Morita equivalence of two pomonoids are (up to natural isomorphism) the tensor multiplication functors. Morita III gives a connection between isomorphism classes of equivalence functors and isomorphism classes of biposets with certain properties.

In this paper, $S$ and $T$ will stand for pomonoids. A poset $(A,\leqslant)$ together with a mapping $A\times S\to A, (a,s)\mapsto a\cdot s$, is called a **right** $S$-**poset** (and the notation $A_S$ is used) if (1) $a\cdot ss'=(a\cdot s)\cdot s'$, (2) $a\cdot 1=a$, (3) $a\leqslant b$ implies $a\cdot s\leqslant b\cdot s$, and (4) $s\leqslant s'$ implies $a\cdot s\leqslant a\cdot s'$, for all $a,b\in A$, $s,s'\in S$. Left $S$-posets can be defined analogously. A left $T$-poset and right $S$-poset $A$ is called a $(T,S)$-**biposet** (and denoted ${}_TA_S$) if $(t\cdot a)\cdot s=t\cdot(a\cdot s)$ for all $a\in A$, $t\in T$ and $s\in S$. By $\mathsf{Pos}_S$ (${}_S\mathsf{Pos}$, ${}_T\mathsf{Pos}_S$) we denote the category of right $S$-posets (resp. left $S$-posets, $(T,S)$-biposets), where the morphisms are order and monoid action preserving mappings. These categories are enriched over the category Pos of posets (with order preserving mappings as morphisms), that is, the morphism sets are posets with respect to pointwise order. A Pos-**functor** between such categories is a functor that preserves the order of morphisms.

Recall that epimorphisms in $\mathsf{Pos}_S$ are surjective morphisms, monomorphisms are injective morphisms, and regular monomorphisms are order embeddings (see Theorem 7 of [3]). It is clear that every coretraction (that is, a left invertible morphism) in $\mathsf{Pos}_S$ is a regular monomorphism.

For a fixed element $a\in A_S$, the mapping $l_a:S\to A,\ s\mapsto a\cdot s$, is a morphism in $\mathsf{Pos}_S$. For fixed elements $s\in S, t\in T$, and ${}_SA_T\in {}_S\mathsf{Pos}_T$, the mappings $\rho_t:A\to A,\ a\mapsto a\cdot t$, and $\lambda_s:A\to A,\ a\mapsto s\cdot a$, are morphisms in ${}_S\mathsf{Pos}$ and $\mathsf{Pos}_T$, respectively.

**Definition 1.** *Pomonoids $S$ and $T$ are called* ***Morita equivalent*** *if the categories $\mathsf{Pos}_S$ and $\mathsf{Pos}_T$ are $\mathsf{Pos}$-equivalent.*

The following lemma is easy to verify.

**Lemma 1.** *For every ${}_SA_T \in {}_S\mathsf{Pos}_T$ there is an isomorphism $S \otimes A \cong A$ in ${}_S\mathsf{Pos}_T$, natural in $A$.*

An object $A_S$ in the category $\mathsf{Pos}_S$ is a **generator** if the functor $\mathsf{Pos}_S(A,-) : \mathsf{Pos}_S \to \mathsf{Pos}$ is faithful. The following results are proved in [6].

**Theorem 1.** *The following assertions are equivalent for a right $S$-poset $A_S$:*
1. *$A_S$ is a generator.*
2. *There exists an epimorphism $\pi : A_S \to S_S$.*
3. *$S_S$ is a retract of $A_S$.*

**Proposition 1.** *Cyclic projectives in $\mathsf{Pos}_S$ are precisely retracts of $S_S$.*

**Proposition 2.** *An $S$-poset $A_S$ is a cyclic projective generator in $\mathsf{Pos}_S$ if and only if $A_S \cong eS_S$ for an idempotent $e \in S$ with $e \mathscr{J} 1$.*

For every $A_T \in \mathsf{Pos}_T$ we consider the set $\mathsf{End}(A_T) = \mathsf{Pos}_T(A,A)$ as a pomonoid with respect to composition and pointwise order. For every ${}_SA \in {}_S\mathsf{Pos}$ we consider the set $\mathsf{End}({}_SA) = {}_S\mathsf{Pos}(A,A)$ as a pomonoid with multiplication $f \bullet g := g \circ f$, $f,g \in \mathsf{End}({}_SA)$, and pointwise order.

**Proposition 3.** *For every ${}_SA_T \in {}_S\mathsf{Pos}_T$, the mappings*

$$\begin{aligned} \lambda &: S \to \mathsf{End}(A_T), \quad s \mapsto \lambda_s, \\ \rho &: T \to \mathsf{End}({}_SA), \quad t \mapsto \rho_t, \end{aligned}$$

*are pomonoid homomorphisms.*

**Definition 2.** *We call a biposet ${}_SA_T$* ***faithfully balanced*** *if the pomonoid homomorphisms $\lambda : S \to \mathsf{End}(A_T)$ and $\rho : T \to \mathsf{End}({}_SA)$ are isomorphisms.*

**Proposition 4.** *Let ${}_SA_T \in {}_S\mathsf{Pos}_T$ be a faithfully balanced biposet. Then $A_T$ is a generator if and only if ${}_SA$ is a cyclic projective.*

**Lemma 2.** *Let ${}_SA_T \in {}_S\mathsf{Pos}_T$. If $A_T$ is a cyclic projective generator and $\lambda : S \to \mathsf{End}(A_T)$ is an isomorphism then ${}_SA_T$ is faithfully balanced.*

## 2. $\mathsf{Pos}$-EQUIVALENCE FUNCTORS

In this section we derive Morita II from a general theorem of [10] about Morita equivalence of enriched categories. Theorem 2 below will use the structures defined in the following lemma.

**Lemma 3.**
1. (a) *For every ${}_SA_T \in {}_S\mathsf{Pos}_T$ and $C_T \in \mathsf{Pos}_T$, the set $\mathsf{Pos}_T(A,C)$ can be considered as an object of $\mathsf{Pos}_S$ with the action defined by*
$$(f \cdot s)(a) := f(s \cdot a). \tag{1}$$
*In particular, the set $\mathsf{Pos}_T(A,T)$ can be considered as an object of ${}_T\mathsf{Pos}_S$ with the actions defined by* (1) *and*
$$(t \cdot f)(a) := tf(a). \tag{2}$$
   (b) *For every ${}_SA_T \in {}_S\mathsf{Pos}_T$ the assignment $C \mapsto \mathsf{Pos}_T(A,C)$ defines a covariant $\mathsf{Pos}$-functor $\mathsf{Pos}_T(A,-) : \mathsf{Pos}_T \to \mathsf{Pos}_S$.*
   (c) *The mapping $\mathsf{Pos}_T(T,T) \to T$, $f \mapsto f(1)$, where the left and right $T$-action on $\mathsf{Pos}_T(T,T)$ are defined by* (1) *and* (2), *is an isomorphism in ${}_T\mathsf{Pos}_T$.*

2. (a) *For every* ${}_TB_S \in {}_T\mathsf{Pos}_S$ *and* ${}_TC \in {}_T\mathsf{Pos}$, *the set* ${}_T\mathsf{Pos}(B,C)$ *can be considered as an object of* ${}_S\mathsf{Pos}$ *with the S-action defined by*
$$(s\cdot f)(b) := f(b\cdot s).$$
   (b) *For every* ${}_TB_S \in {}_T\mathsf{Pos}_S$ *the assignment* $C \mapsto {}_T\mathsf{Pos}(B,C)$ *defines a covariant* Pos-*functor* ${}_T\mathsf{Pos}(B,-) : {}_T\mathsf{Pos} \to {}_S\mathsf{Pos}$.

In the notation of [10] (Def. 2.6), $\mathsf{Pos}_T(A,-) : \mathsf{Pos}_T \to \mathsf{Pos}_S$ is the functor $A^\vee$.

**Definition 3.** *An* $(S,T)$*-biposet* ${}_SP_T$ *is called* Pos-***prodense*** (*see Theorem* 2.8 *of* [10]) *if the functor* $\mathsf{Pos}_T(P,-) : \mathsf{Pos}_T \to \mathsf{Pos}_S$ *is a* Pos-*equivalence*.

For the details about tensor products of $S$-posets we refer to [12]. As in [10], by a Pos-adjoint we mean a Pos-functor that has a left adjoint functor which is also a Pos-functor. A Pos-cocontinuous functor is a Pos-functor that preserves all small Pos-colimits. Theorem 3.11 of [10], specified for pomonoids (one-object Pos-categories), gives the following.

**Theorem 2.** *Let* $S,T$ *be pomonoids.*
(a) *If* $F : \mathsf{Pos}_S \to \mathsf{Pos}_T$ *is a* Pos-*adjoint functor*, *then there exists a biposet* ${}_TQ_S$ *such that* $F \cong \mathsf{Pos}_S(Q,-)$.
(b) *If* $F : \mathsf{Pos}_S \to \mathsf{Pos}_T$ *is a* Pos-*cocontinuous functor*, *then there exists a biposet* ${}_SP_T$ *such that* $F \cong -\otimes_S P$.
(c) *If* $F : \mathsf{Pos}_S \to \mathsf{Pos}_T$ *is* Pos-*adjoint*, Pos-*cocontinuous*, *and* Pos-*fully faithful*, *then* ${}_S(P\otimes_T Q)_S \cong {}_SS_S$, *where* ${}_SP_T, {}_TQ_S$ *are as in* (a) *and* (b).
(d) *Let* $F : \mathsf{Pos}_S \to \mathsf{Pos}_T$ *be a* Pos-*equivalence and let* ${}_SP_T, {}_TQ_S$ *be as in* (a) *and* (b).
   (i) *The functor* $-\otimes_T Q : \mathsf{Pos}_T \to \mathsf{Pos}_S$ *is a* Pos-*equivalence inverse of* $-\otimes_S P$ *and* $\mathsf{Pos}_S(Q,-)$. *Furthermore*,
$${}_T(Q\otimes_S P)_T \cong {}_TT_T, \;\; {}_SP_T \cong \mathsf{Pos}_S(Q,S), \;\; {}_TQ_S \cong \mathsf{Pos}_T(P,T).$$
   (ii) *The functor* $Q\otimes_S - : {}_S\mathsf{Pos} \to {}_T\mathsf{Pos}$ *is a* Pos-*equivalence with inverses* $P\otimes_T -$ *and* ${}_T\mathsf{Pos}(Q,-)$.
(e) *If a biposet* ${}_SP_T$ *is* Pos-*prodense*, *then the functor* ${}_S\mathsf{Pos}(P,-) : {}_S\mathsf{Pos} \to {}_T\mathsf{Pos}$ *is a* Pos-*equivalence*.

This gives us a necessary and sufficient condition for Morita equivalence of two pomonoids.

**Corollary 1.** *Pomonoids S and T are Morita equivalent if and only if there exists a* Pos-*prodense biposet* ${}_SP_T$.

*Proof. Necessity*. Let $G : \mathsf{Pos}_T \to \mathsf{Pos}_S$ be a Pos-equivalence functor. By Theorem 2(a), there exists a biposet ${}_SP_T$ such that $G \cong \mathsf{Pos}_T(P,-)$, hence also $\mathsf{Pos}_T(P,-)$ is a Pos-equivalence and ${}_SP_T$ is Pos-prodense.
*Sufficiency* is clear. □

Let us give some more conditions for Morita equivalence of two pomonoids. By $\mathsf{CPG}_S$ we denote the full subcategory of $\mathsf{Pos}_S$ generated by all cyclic projective generators. We say that a posemigroup $S$ is an **enlargement** of a posemigroup $T$ (cf. [8]) if $T$ is isomorphic to a subposemigroup $S'$ of $S$ such that $S = SS'S$ and $S' = S'SS'$.

**Theorem 3.** *The following assertions are equivalent for pomonoids S and T.*
1. *S and T are Morita equivalent.*
2. *The categories* $\mathsf{CPG}_S$ *and* $\mathsf{CPG}_T$ *are* Pos-*equivalent.*
3. *There exists* $Q_S \in \mathsf{CPG}_S$ *such that* $T \cong \mathsf{End}(Q_S)$ *as pomonoids.*
4. *There exists an idempotent* $e \in S$ *such that* $e\,\mathscr{J}\,1$ *and* $T \cong eSe$ *as pomonoids.*
5. *S is an enlargement of T.*

*Proof.* 1. ⇒ 2. It is not difficult to see that Pos-equivalence functors between $\mathsf{Pos}_S$ and $\mathsf{Pos}_T$ take cyclic projective generators to cyclic projective generators. Hence they induce a Pos-equivalence between $\mathsf{CPG}_S$ and $\mathsf{CPG}_T$.

2. ⇒ 3. Suppose that $\mathsf{CPG}_S \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{CPG}_T$ are mutually inverse Pos-equivalence functors and denote $Q_S := G(T) \in \mathsf{CPG}_S$. Then $T \cong \mathsf{End}(T_T) \cong \mathsf{End}(Q_S)$ as pomonoids.

3. $\Rightarrow$ 4. Since $Q_S$ is a cyclic projective generator, by Proposition 2 there exists an idempotent $e \in S$ such that $e \mathscr{J} 1$ and $Q \cong eS$ in $\mathsf{Pos}_S$. Hence

$$T \cong \mathsf{End}(Q_S) \cong \mathsf{End}(eS_S) \cong eSe$$

as pomonoids, where an isomorphism $\varphi : \mathsf{End}(eS_S) = \mathsf{Pos}_S(eS, eS) \to eSe$ is defined by

$$\varphi(f) := f(e)$$

(cf. Proposition 1.5.6 of [4]).

4. $\Rightarrow$ 5. Let $T \cong eSe$, where $e \in S$ is an idempotent and $kel = 1$, $k, l \in S$. The equality $eSe = (eSe)S(eSe)$ is obvious. The equality $S = S(eSe)S$ holds because $s = kelskel$ for every $s \in S$. Hence $S$ is an enlargement of $T$.

5. $\Rightarrow$ 4. Suppose that $S'$ is a subposemigroup of $S$ such that $S = SS'S$ and $S' = S'SS'$, and there is an isomorphism $\varphi : T \to S'$ of posemigroups. Then $e = \varphi(1)$ is the identity element for $S'$. Consequently, $S' = eS'e \subseteq eSe$, but also $eSe \subseteq S'SS' = S'$. Thus $S' = eSe$ and $\varphi : T \to eSe$ is a pomonoid isomorphism. In addition, $1 = s_1 s' s_2 = s_1 s' e s_2$ in $S$ for some $s_1, s_2 \in S$, $s' \in S'$.

4. $\Rightarrow$ 1. Let $e \in S$ be an idempotent such that $e \mathscr{J} 1$ and $T \cong eSe$. It suffices to prove that $\mathsf{Pos}_S$ and $\mathsf{Pos}_{eSe}$ are $\mathsf{Pos}$-equivalent categories. If $A_S \in \mathsf{Pos}_S$ then the set $Ae := \{a \cdot e \mid a \in A\}$ can be considered as a right $eSe$-poset with the action $(a \cdot e, ese) \mapsto a \cdot ese$. We define a $\mathsf{Pos}$-functor $F : \mathsf{Pos}_S \to \mathsf{Pos}_{eSe}$ by the assignment

$$\begin{tikzcd} A_S \arrow[r, mapsto] \arrow[d, "g"'] & Ae_{eSe} \arrow[d, "\overline{g}"] \\ B_S \arrow[r, mapsto] & Be_{eSe} \end{tikzcd},$$

where $\overline{g} : a \cdot e \mapsto g(a \cdot e) = g(a) \cdot e \in Be$. Similarly to the unordered case (see Proposition 5.3.12 of [4]), one can show that $F$ is a $\mathsf{Pos}$-equivalence functor. $\square$

**Remark 1.** One can see that pomonoids $S$ and $T$ are Morita equivalent if and only if ${}_S\mathsf{Pos}$ and ${}_T\mathsf{Pos}$ are $\mathsf{Pos}$-equivalent categories by noting that cyclic projective generators in ${}_S\mathsf{Pos}$ are of the form $Se$ where $e \mathscr{J} 1$, and using a proof that is similar to the proof of Theorem 3.

Also, $S$ is an enlargement of $T$ if and only if $S$ and $T$ are enlargements of each other if and only if $S$ and $T$ have a joint enlargement. This way Theorem 3 can be compared to Theorem 1.1 of [9].

Theorem 3 shows that being Morita equivalent is in the case of pomonoids very close to being isomorphic. As in the monoid case (see [4], Corollary 5.3.14, or [2], corollary to Proposition 4), for several large classes of pomonoids these notions coincide.

**Corollary 2.** *Morita equivalence of the pomonoids $S$ and $T$ implies that $S$ and $T$ are isomorphic pomonoids whenever* 1 *is the only idempotent in its* $\mathscr{J}$*-class. In particular, this is true in either of the following cases*:
1. *$S$ has central idempotents*;
2. *every right invertible element of $S$ is left invertible or vice versa*;
3. *all elements of infinite order in $S$ are powers of one element*;
4. *idempotents of $S$ satisfy the ascending chain condition*;
5. *$S$ satisfies the descending chain condition for principal right (or left) ideals.*

A list of non-isomorphic Morita equivalent monoids (which can be regarded as trivially ordered pomonoids) is given in [4]. We give here an example of non-isomorphic Morita equivalent pomonoids with non-trivial order. This will be a modification of Example 7.1 from [5].

**Example 1.** Consider the real interval $[0, 1]$ and the monoid

$$S' = \{f : [0, x] \to [0, 1] \mid x \in [0, 1], f \text{ is strictly increasing and continuous}\} \cup \{\emptyset : \emptyset \to [0, 1]\}$$

with the multiplication

$$gf : \{a \in \operatorname{dom} f \mid f(a) \in \operatorname{dom} g\} \to [0,1], \quad a \mapsto g(f(a)),$$

and order relation

$$f \leqslant h \Longleftrightarrow \operatorname{dom} f \subseteq \operatorname{dom} h \wedge (\forall a \in \operatorname{dom} f)(f(a) \geqslant h(a)).$$

Note that if $\operatorname{dom} f = [0,x]$ and $\operatorname{dom} g = [0,y]$, then

$$\operatorname{dom}(gf) = \begin{cases} [0, \max\{a \in [0,x] \mid f(a) \leqslant y\}], & \text{if } f(0) \leqslant y, \\ \emptyset, & \text{if } f(0) > y, \end{cases}$$

so, indeed, $gf \in S'$.

Let us check that $S'$ is a pomonoid. Suppose $f,g,h \in S'$ and $f \leqslant h$. To prove that $gf \leqslant gh$, we first have to show that $\operatorname{dom}(gf) \subseteq \operatorname{dom}(gh)$. If $a \in \operatorname{dom}(gf)$, then $a \in \operatorname{dom} f \subseteq \operatorname{dom} h$ and $f(a) \in \operatorname{dom} g$. Therefore $h(a) \leqslant f(a) \in \operatorname{dom} g$. Since $\operatorname{dom} g$ is a down-set in the poset $[0,1]$, also $h(a) \in \operatorname{dom} g$, and hence $a \in \operatorname{dom}(gh)$. Thus $\operatorname{dom}(gf) \subseteq \operatorname{dom}(gh)$. For every $a \in \operatorname{dom} f$ we have $f(a) \geqslant h(a)$. Since $g$ preserves order and $\operatorname{dom}(gf) \subseteq \operatorname{dom} f$, we also have $(gf)(a) \geqslant (gh)(a)$ for every $a \in \operatorname{dom}(gf)$. Consequently, $gf \leqslant gh$.

To verify the inequality $fg \leqslant hg$ we notice that the inclusion $\operatorname{dom}(fg) \subseteq \operatorname{dom}(hg)$ follows from the inclusion $\operatorname{dom} f \subseteq \operatorname{dom} h$. If $a \in \operatorname{dom}(fg) = \{b \in \operatorname{dom} g \mid g(b) \in \operatorname{dom} f\}$, then $f(g(a)) \geqslant h(g(a))$. Hence $fg \leqslant hg$.

For every $x \in [0,1]$ let $i_x : [0,x] \to [0,1]$, $a \mapsto a$, and consider also the mappings

$$\begin{aligned} k:&\quad \left[0,\tfrac{1}{2}\right] \to [0,1], \quad a \mapsto 2a, \\ l:&\quad [0,1] \to [0,1], \quad a \mapsto \tfrac{a}{2}. \end{aligned}$$

Note that $k \leqslant l$ and $i_x \leqslant i_y$ if and only if $x \leqslant y$. Let $S$ be the subpomonoid of $S'$ generated by the set

$$\{k,l\} \cup \left\{ i_x \mid x \in \left[0, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right] \right\}.$$

It is easy to see that $i_{\frac{3}{4}}$ is an idempotent in $S$ and $ki_{\frac{3}{4}}l = i_1$, where $i_1 = 1_{[0,1]}$ is the identity element of $S$. Thus $S$ is Morita equivalent to $i_{\frac{3}{4}}Si_{\frac{3}{4}}$.

We claim that $S$ and $i_{\frac{3}{4}}Si_{\frac{3}{4}}$ are not isomorphic pomonoids. It can be seen that the idempotents of $S$ are $i_x$, where $x \in \left[0,\frac{1}{2}\right] \cup \left[\frac{3}{4},1\right]$. So the idempotents of $S$ that are different from the identity element $i_1$ form a chain that contains no supremum. But the idempotents of $i_{\frac{3}{4}}Si_{\frac{3}{4}}$ are $i_x$ where $x \in \left[0,\frac{1}{2}\right] \cup \left\{\frac{3}{4}\right\}$. Thus the chain of non-identity idempotents of $i_{\frac{3}{4}}Si_{\frac{3}{4}}$ has the supremum $i_{\frac{1}{2}}$. Hence $S$ and $i_{\frac{3}{4}}Si_{\frac{3}{4}}$ cannot be isomorphic pomonoids, because a pomonoid isomorphism induces an isomorphism between the posets of idempotents.

For the next theorem we shall need the following lemma.

**Lemma 4.** *Let ${}_SP_T, {}_SP'_T \in {}_S\mathsf{Pos}_T$. The functors $-\otimes_S P, -\otimes_S P' : \mathsf{Pos}_S \to \mathsf{Pos}_T$ are naturally isomorphic if and only if $P \cong P'$ in ${}_S\mathsf{Pos}_T$.*

*Proof. Necessity.* Suppose that $\alpha : -\otimes_S P \to -\otimes_S P'$ is a natural isomorphism. Then $\alpha_S : S \otimes P \to S \otimes P'$ is an isomorphism in $\mathsf{Pos}_T$. Due to Lemma 1, we only need to check that $\alpha_S$ is a morphism of left $S$-posets. To this end, take any $s,s' \in S$ and $p \in P$. Since $l_{s'} : S \to S$, $z \mapsto s'z$, is a morphism in $\mathsf{Pos}_S$ and $\alpha$ is a natural transformation, the square

$$\begin{tikzcd} S \otimes P \arrow[r, "\alpha_S"] \arrow[d, "l_{s'} \otimes 1_P"'] & S \otimes P' \arrow[d, "l_{s'} \otimes 1_{P'}"] \\ S \otimes P \arrow[r, "\alpha_S"'] & S \otimes P' \end{tikzcd}$$

commutes in $\mathsf{Pos}_T$. Note that $(l_{s'} \otimes 1_{P'})(s'' \otimes p') = s's'' \otimes p' = s' \cdot (s'' \otimes p')$ for all $s'' \in S$ and $p' \in P'$, so $l_{s'} \otimes 1_{P'} = \lambda_{s'}$, where $\lambda_{s'} : S \otimes P' \to S \otimes P'$, $x \mapsto s'x$. Hence

$$\begin{aligned}\alpha_S(s' \cdot (s \otimes p)) &= \alpha_S(s's \otimes p) = (\alpha_S(l_{s'} \otimes 1_P))(s \otimes p) \\ &= ((l_{s'} \otimes 1_{P'})\alpha_S)(s \otimes p) = \lambda_{s'}(\alpha_S(s \otimes p)) = s' \cdot \alpha_S(s \otimes p).\end{aligned}$$

*Sufficiency*. Let $\varphi : P \to P'$ be an isomorphism in ${}_S\mathsf{Pos}_T$. If $A_S \in \mathsf{Pos}_S$, then the functor $A \otimes_S - : {}_S\mathsf{Pos}_T \to \mathsf{Pos}_T$ takes $\varphi$ to an isomorphism $1_A \otimes \varphi : A \otimes P \to A \otimes P'$ in $\mathsf{Pos}_T$. If $f : A \to B$ is any morphism in $\mathsf{Pos}_S$, then obviously the square

$$\begin{tikzcd} A \otimes P \arrow[r, "1_A \otimes \varphi"] \arrow[d, "f \otimes 1_P"'] & A \otimes P' \arrow[d, "f \otimes 1_{P'}"] \\ B \otimes P \arrow[r, "1_B \otimes \varphi"'] & B \otimes P' \end{tikzcd}$$

commutes and hence $(1_A \otimes \varphi)_{A \in \mathsf{Pos}_S} : - \otimes P \to - \otimes P'$ is a natural isomorphism. □

Now we can prove a theorem that corresponds to Morita II in the case of pomonoids.

**Theorem 4** (Morita II). *Let $S, T$ be pomonoids and let $\mathsf{Pos}_S \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{Pos}_T$ be mutually inverse $\mathsf{Pos}$-equivalence functors. Then $P := F(S) \in {}_S\mathsf{Pos}_T$, $Q := G(T) \in {}_T\mathsf{Pos}_S$ and*

$$F \cong - \otimes_S P, \qquad G \cong - \otimes_T Q.$$

*Proof*. If $\mathsf{Pos}_S \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{Pos}_T$ are mutually inverse Pos-equivalence functors, then $F(S)$ can be considered as an object of ${}_S\mathsf{Pos}_T$ with the left $S$-action defined by

$$s \cdot b := F(l_s)(b)$$

for every $b \in F(S)$. Indeed, it is known (see Lemma 5.3.1 of [4]) that such $F(S)$ will be an $(S,T)$-biact. Suppose that $s \leqslant z$, $s, z \in S$. Then $l_s \leqslant l_z$, hence $F(l_s) \leqslant F(l_z)$ and $s \cdot b = F(l_s)(b) \leqslant F(l_z)(b) = z \cdot b$ for every $b \in F(S)$. If $b \leqslant c$, $b, c \in F(S)$, and $s \in S$, then $s \cdot b = F(l_s)(b) \leqslant F(l_s)(c) = s \cdot c$ because $F(l_s)$ is a morphism in $\mathsf{Pos}_T$. Hence $F(S) \in {}_S\mathsf{Pos}_T$.

By Theorem 2(b), there exist a biposet ${}_SP'_T$ and a natural isomorphism $\alpha : F \to - \otimes_S P'$. As in the proof of Lemma 4, $l_{s'} \otimes 1_{P'} = \lambda_{s'}$, and so, by naturality,

$$\begin{aligned}\alpha_S(s' \cdot b) &= \alpha_S(F(l_{s'})(b)) = (\alpha_S F(l_{s'}))(b) = ((l_{s'} \otimes 1_{P'})\alpha_S)(b) = \lambda_{s'}(\alpha_S(b)) \\ &= s' \cdot \alpha_S(b)\end{aligned}$$

for every $s' \in S$, $b \in F(S)$. This means that $\alpha_S : F(S) \to S \otimes P'$ is a morphism in ${}_S\mathsf{Pos}$ and hence an isomorphism in ${}_S\mathsf{Pos}_T$. By Lemma 1, ${}_SP_T = {}_SF(S)_T \cong {}_S(S \otimes P')_T \cong {}_SP'_T$ in ${}_S\mathsf{Pos}_T$, and by Lemma 4, $F \cong - \otimes_S P' \cong - \otimes_S P$. Similarly, $G \cong - \otimes_T Q$. □

## 3. Pos-PRODENCE BIPOSETS

Here we give a description of Pos-prodense objects of ${}_S\mathsf{Pos}_T$, which, as we have seen in the previous section, play an important role in Morita theory. First we prove some technical results.

**Proposition 5.** *If ${}_SP_T \in {}_S\mathsf{Pos}_T$ is such that $P_T$ is a cyclic projective, then $P \otimes_T \mathsf{Pos}_T(P,T) \cong \mathsf{Pos}_T(P,P)$ in ${}_S\mathsf{Pos}_S$.*

*Proof.* Note that the right $S$-action on $\mathsf{Pos}_T(P,P) \in \mathsf{Pos}_S$ is defined by $(f \cdot s)(p) := f(s \cdot p)$ (see (1)) and the actions on $\mathsf{Pos}_T(P,T) \in {}_T\mathsf{Pos}_S$ are defined in Lemma 3(1). We define a mapping $\mu : P \otimes_T \mathsf{Pos}_T(P,T) \longrightarrow \mathsf{Pos}_T(P,P)$ by

$$\mu(a \otimes f) := a \cdot f(-),$$

$a \in P$, $f \in \mathsf{Pos}_T(P,T)$. Since

$$\mu(a \otimes f)(p \cdot t) = a \cdot f(p \cdot t) = a \cdot (f(p)t) = (a \cdot f(p)) \cdot t = (\mu(a \otimes f)(p)) \cdot t$$

for all $a, p \in P$, $f \in \mathsf{Pos}_T(P,T)$, $t \in T$, and since $\mu(a \otimes f) : P \to P$ obviously preserves order, it is a morphism in $\mathsf{Pos}_T$.

Let us prove that $\mu$ preserves order. Suppose that $a \otimes f \leqslant a' \otimes f'$ in $P \otimes \mathsf{Pos}_T(P,T)$, $a, a' \in P$, $f, f' \in \mathsf{Pos}_T(P,T)$. Then there exist a natural number $n$ and $a_1, \ldots, a_n \in A$, $f_2, \ldots, f_n \in \mathsf{Pos}_T(P,T)$, $t_1, \ldots, t_n, u_1, \ldots, u_n \in T$ such that

$$\begin{array}{rclrcl}
a & \leqslant & a_1 \cdot t_1 & & & \\
a_1 \cdot u_1 & \leqslant & a_2 \cdot t_2 & t_1 \cdot f & \leqslant & u_1 \cdot f_2 \\
a_2 \cdot u_2 & \leqslant & a_3 \cdot t_3 & t_2 \cdot f_2 & \leqslant & u_2 \cdot f_3 \\
& \cdots & & & \cdots & \\
a_n \cdot u_n & \leqslant & a' & t_n \cdot f_n & \leqslant & u_n \cdot f'.
\end{array}$$

Applying the morphisms of the right hand side column to an element $p \in P$ we obtain

$$\begin{array}{rclrcl}
a & \leqslant & a_1 \cdot t_1 & & & \\
a_1 \cdot u_1 & \leqslant & a_2 \cdot t_2 & t_1 f(p) & \leqslant & u_1 f_2(p) \\
a_2 \cdot u_2 & \leqslant & a_3 \cdot t_3 & t_2 f_2(p) & \leqslant & u_2 f_3(p) \\
& \cdots & & & \cdots & \\
a_n \cdot u_n & \leqslant & a' & t_n f_n(p) & \leqslant & u_n f'(p),
\end{array}$$

which implies $\mu(a \otimes f)(p) = a \cdot f(p) \leqslant a' \cdot f'(p) = \mu(a' \otimes f')(p)$ in $P$. In this way we have shown that $\mu(a \otimes f) \leqslant \mu(a' \otimes f')$ in $\mathsf{Pos}_T(P,P)$ (in particular, that $\mu$ is well defined and order preserving).

To prove that $\mu$ is a morphism in ${}_S\mathsf{Pos}_S$ we note that

$$\begin{aligned}
\mu((a \otimes f) \cdot s)(p) &= \mu(a \otimes f \cdot s)(p) = a \cdot (f \cdot s)(p) = a \cdot f(s \cdot p) \\
&= \mu(a \otimes f)(s \cdot p) = (\mu(a \otimes f) \cdot s)(p) = \mu(s \cdot (a \otimes f))(p) \\
&= \mu(s \cdot a \otimes f)(p) = (s \cdot a) \cdot f(p) = s \cdot (a \cdot f(p)) \\
&= s \cdot \mu(a \otimes f)(p)
\end{aligned}$$

for all $a, p \in P$, $f \in \mathsf{Pos}_T(P,T)$, $s \in S$.

By Proposition 1 there exist morphisms $P \underset{\beta}{\overset{\alpha}{\rightleftarrows}} T$ in $\mathsf{Pos}_T$ with $\beta \circ \alpha = 1_P$. To see that $\mu$ is surjective, take $g \in \mathsf{Pos}_T(P,P)$ and denote $a := \beta(1)$, $f := \alpha \circ g$. Then

$$\beta(1)(\alpha(g(p))) = \beta(1\alpha(g(p))) = (\beta\alpha)(g(p)) = g(p)$$

for every $p \in P$ and hence $\mu(a \otimes f) = \beta(1) \cdot (\alpha \circ g)(-) = g$. To prove that $\mu$ reflects order, suppose that $a \cdot f(-) \leqslant a' \cdot f'(-)$, $a, a' \in P$, $f, f' \in \mathsf{Pos}_T(P,T)$. Then $a \cdot f(\beta(1)) \leqslant a' \cdot f'(\beta(1))$. Note that

$$((f \circ \beta)(1) \cdot \alpha)(p) = (f \circ \beta)(1)\alpha(p) = (f \circ \beta)(\alpha(p)) = (f \circ \beta \circ \alpha)(p) = f(p)$$

for every $p \in P$, so $(f \circ \beta)(1) \cdot \alpha = f$, and similarly $(f' \circ \beta)(1) \cdot \alpha = f'$. Consequently,

$$\begin{aligned} a \otimes f &= a \otimes (f \circ \beta)(1) \cdot \alpha = a \cdot (f \circ \beta)(1) \otimes \alpha \leqslant a' \cdot (f' \circ \beta)(1) \otimes \alpha \\ &= a' \otimes (f' \circ \beta)(1) \cdot \alpha = a' \otimes f'. \end{aligned}$$ □

**Lemma 5.** *For every ${}_SP_T \in {}_S\mathsf{Pos}_T$*
1. *the set ${}_S\mathsf{Pos}(P,P)$ can be considered as an object of ${}_T\mathsf{Pos}_T$ with the actions defined by*

$$\begin{aligned} f \cdot t &:= \rho_t \circ f, \qquad (3) \\ t \cdot f &:= f \circ \rho_t, \qquad (4) \end{aligned}$$

$f \in {}_S\mathsf{Pos}(P,P)$, $t \in T$;
2. $\rho : {}_TT_T \to {}_S\mathsf{Pos}(P,P)$ *is a morphism in ${}_T\mathsf{Pos}_T$.*

**Proposition 6.** *If a biposet ${}_SP_T \in {}_S\mathsf{Pos}_T$ is such that ${}_SP$ is a cyclic projective, then* $\mathsf{Pos}_T(P,T) \otimes_S P \cong \mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T)$ *in ${}_T\mathsf{Pos}_T$.*

*Proof.* Note that the right $S$-action on $\mathsf{Pos}_T(P,T) \in {}_T\mathsf{Pos}_S$ is defined by (1) and the left $T$-action by (2), the $T$-actions on ${}_S\mathsf{Pos}(P,P) \in {}_T\mathsf{Pos}_T$ are defined by (3) and (4) in Lemma 5, the right $T$-action on $\mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T) \in \mathsf{Pos}_T$ is defined by $(m \cdot t)(f) := m(t \cdot f) = m(f \circ \rho_t)$ (see again (1)) and the left $T$-action on $\mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T)$ by (2). We define a mapping

$$\nu : \mathsf{Pos}_T(P,T) \otimes_S P \longrightarrow \mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T)$$

by

$$\nu(g \otimes p)(f) := g(f(p)),$$

$g \in \mathsf{Pos}_T(P,T)$, $p \in P$, $f \in {}_S\mathsf{Pos}(P,P)$. First we show that $\nu$ preserves order. Suppose that $g \otimes p \leqslant g' \otimes p'$ in $\mathsf{Pos}_T(P,T) \otimes_S P$, $g, g' \in \mathsf{Pos}_T(P,T)$, $p, p' \in P$. Then

$$\begin{array}{rclcrcl} g & \leqslant & g_1 \cdot s_1 & & & & \\ g_1 \cdot z_1 & \leqslant & g_2 \cdot s_2 & & s_1 \cdot p & \leqslant & z_1 \cdot p_2 \\ g_2 \cdot z_2 & \leqslant & g_3 \cdot s_3 & & s_2 \cdot p_2 & \leqslant & z_2 \cdot p_3 \\ & \cdots & & & & \cdots & \\ g_n \cdot z_n & \leqslant & g' & & s_n \cdot p_n & \leqslant & z_n \cdot p' \end{array}$$

for some $g_1, \dots, g_n \in \mathsf{Pos}_T(P,T)$, $p_2, \dots, p_n \in P$, $s_1, \dots, s_n, z_1, \dots, z_n \in S$. Using these inequalities, for every $f \in {}_S\mathsf{Pos}(P,P)$ we have

$$\begin{aligned} \nu(g \otimes p)(f) &= g(f(p)) \leqslant (g_1 \cdot s_1)(f(p)) = g_1(s_1 \cdot f(p)) = g_1(f(s_1 \cdot p)) \\ &\leqslant g_1(f(z_1 \cdot p_2)) = g_1(z_1 \cdot f(p_2)) = (g_1 \cdot z_1)(f(p_2)) \leqslant \dots \\ &\leqslant (g_n \cdot z_n)(f(p')) \leqslant g'(f(p')) = \nu(g' \otimes p')(f), \end{aligned}$$

and hence $\nu$ is order preserving (therefore also well defined).

Next we prove that $\nu(g \otimes p) : {}_S\mathsf{Pos}(P,P) \to T$ is a morphism in $\mathsf{Pos}_T$. Indeed,

$$\nu(g \otimes p)(f \cdot t) = g((f \cdot t)(p)) = g((\rho_t \circ f)(p)) = g(f(p) \cdot t) = g(f(p)) \cdot t = (\nu(g \otimes p)(f)) \cdot t$$

for all $g \in \mathsf{Pos}_T(P,T)$, $p \in P$, $f \in {}_S\mathsf{Pos}(P,P)$, $t \in T$, and obviously $\nu(g \otimes p)$ preserves order. Also

$$\begin{aligned} \nu((g \otimes p) \cdot t)(f) &= \nu(g \otimes p \cdot t)(f) = g(f(p \cdot t)) = g((f \circ \rho_t)(p)) \\ &= \nu(g \otimes p)(t \cdot f) = (\nu(g \otimes p) \cdot t)(f), \\ \nu(t \cdot (g \otimes p))(f) &= \nu(t \cdot g \otimes p)(f) = (t \cdot g)(f(p)) = tg(f(p)) = t(\nu(g \otimes p)(f)) \\ &= (t \cdot \nu(g \otimes p))(f) \end{aligned}$$

for all $g \in \mathsf{Pos}_T(P,T)$, $p \in P$, $t \in T$, $f \in {}_S\mathsf{Pos}(P,P)$, and hence $\nu$ is a morphism in ${}_T\mathsf{Pos}_T$.

Since ${}_SP$ is a cyclic projective, by the dual of Proposition 1 there exist morphisms $P \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S$ in ${}_S\mathsf{Pos}$ with $\beta \circ \alpha = 1_P$. By the dual of Lemma 1 and the proof of Lemma 2 (1c) of [6], the morphisms

$$\begin{aligned} \psi &: \mathsf{Pos}_T(P,T) \otimes_S S \longrightarrow \mathsf{Pos}_T(P,T),\ g \otimes s \mapsto g \cdot s, \text{ in } \mathsf{Pos}_S, \\ \varphi &: {}_S\mathsf{Pos}(S,P) \longrightarrow P,\ u \mapsto u(1), \text{ in } \mathsf{Pos}_T, \end{aligned}$$

are isomorphisms. Hence also

$$\mathsf{Pos}_T(-,T)(\varphi) = - \circ \varphi : \mathsf{Pos}_T(P,T) \longrightarrow \mathsf{Pos}_T({}_S\mathsf{Pos}(S,P),T)$$

and the composite

$$\nu_S = (- \circ \varphi) \circ \psi : \mathsf{Pos}_T(P,T) \otimes_S S \longrightarrow \mathsf{Pos}_T({}_S\mathsf{Pos}(S,P),T)$$

are isomorphisms in Pos. Note that

$$\nu_S(g \otimes s)(u) = (\psi(g \otimes s) \circ \varphi)(u) = \psi(g \otimes s)(u(1)) = (g \cdot s)(u(1)) = g(s \cdot u(1)) = g(u(s))$$

for all $g \in \mathsf{Pos}_T(P,T)$, $s \in S$, $u \in {}_S\mathsf{Pos}(S,P)$. Since

$$\begin{aligned} (((- \circ (- \circ \beta)) \circ \nu_S)(g \otimes s))(f) &= (\nu_S(g \otimes s) \circ (- \circ \beta))(f) \\ &= \nu_S(g \otimes s)(f \circ \beta) = g((f \circ \beta)(s)) \\ &= g(f(\beta(s))) = \nu(g \otimes \beta(s))(f) \\ &= ((\nu \circ (1 \otimes \beta))(g \otimes s))(f), \\ ((\nu_S \circ (1 \otimes \alpha))(g \otimes p))(u) &= \nu_S(g \otimes \alpha(p))(u) = g(u(\alpha(p))) \\ &= \nu(g \otimes p)(u \circ \alpha) = (\nu(g \otimes p) \circ (- \circ \alpha))(u) \\ &= (((- \circ (- \circ \alpha)) \circ \nu)(g \otimes p))(u) \end{aligned}$$

for all $g \in \mathsf{Pos}_T(P,T)$, $s \in S$, $f \in {}_S\mathsf{Pos}(P,P)$, $u \in {}_S\mathsf{Pos}(S,P)$, the left hand square and the right hand square in the diagram

$$\begin{tikzcd} \mathsf{Pos}_T(P,T) \otimes_S S \arrow[r, "\nu_S"] \arrow[d, shift right, "1\otimes\beta"'] & \mathsf{Pos}_T({}_S\mathsf{Pos}(S,P),T) \arrow[d, shift right, "-\circ(-\circ\beta)"'] \\ \mathsf{Pos}_T(P,T) \otimes_S P \arrow[r, "\nu"] \arrow[u, shift right, "1\otimes\alpha"'] & \mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T) \arrow[u, shift right, "-\circ(-\circ\alpha)"'] \end{tikzcd}$$

commute. The equality $\beta \circ \alpha = 1_P$ implies $(1 \otimes \beta) \circ (1 \otimes \alpha) = 1_{\mathsf{Pos}_T(P,T) \otimes P}$ and $(- \circ (- \circ \beta)) \circ (- \circ (- \circ \alpha)) = 1_{\mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T)}$. Then $\nu$ is a retraction in Pos, because

$$\nu \circ (1 \otimes \beta) \circ \nu_S^{-1} \circ (- \circ (- \circ \alpha)) = (- \circ (- \circ \beta)) \circ \nu_S \circ \nu_S^{-1} \circ (- \circ (- \circ \alpha)) = 1_{\mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T)},$$

and similarly it is a coretraction. Therefore it is an isomorphism in ${}_T\mathsf{Pos}_T$. □

There is an isomorphism between the category ${}_S\mathsf{Pos}_T$ and the category of contravariant Pos-functors $\mathbf{1} \to \mathsf{Pos}_T$, where $\mathbf{1}$ is the category with one object $*$, $\mathbf{1}(*,*) = S$, and the composition in $\mathbf{1}$ is given by the multiplication in $S$. The Pos-functor $\mathbf{P} : \mathbf{1} \to \mathsf{Pos}_T$ corresponding to a biposet ${}_SP_T$ is given by the assignment

$$\begin{tikzcd} * \arrow[r, mapsto] \arrow[d, "s"'] & P \arrow[d, "\lambda_s"] \\ * \arrow[r, mapsto] & P \end{tikzcd} .$$

The following lemma is easy to verify.

**Lemma 6.** *If a biposet ${}_SP_T$ is* $\mathsf{Pos}$*-prodense and* ${}_SP_T \cong {}_SQ_T$ *in* ${}_S\mathsf{Pos}_T$*, then also* ${}_SQ_T$ *is* $\mathsf{Pos}$*-prodense.*

**Theorem 5.** *For a biposet* ${}_SP_T \in {}_S\mathsf{Pos}_T$*, the following assertions are equivalent.*
1. ${}_SP_T$ *is* $\mathsf{Pos}$*-prodense.*
2. ${}_SP_T$ *is faithfully balanced and* ${}_SP, P_T$ *are cyclic projective generators.*
3. *There exists a biposet* ${}_TP^*_S \in {}_T\mathsf{Pos}_S$ *such that*

$$\begin{aligned} P \otimes_T P^* &\cong S \quad \text{in} \quad {}_S\mathsf{Pos}_S, \\ P^* \otimes_S P &\cong T \quad \text{in} \quad {}_T\mathsf{Pos}_T. \end{aligned}$$

*Proof.* 1. $\Rightarrow$ 2. Let ${}_SP_T$ be Pos-prodense and consider the pomonoid homomorphism

$$\lambda : S \to \mathsf{Pos}_T(P,P), s \mapsto \lambda_s = \mathbf{P}(s)$$

(see Proposition 3). This morphism is an isomorphism of posets (and hence an isomorphism of pomonoids, but also an isomorphism in ${}_S\mathsf{Pos}_S$ by the dual of Lemma 5) because the functor $\mathbf{P}$ is Pos-fully faithful by Theorem 2.8(e) of [10]. Since the functor $\mathsf{Pos}_T(P,-) : \mathsf{Pos}_T \to \mathsf{Pos}_S$ is faithful, $P_T$ is a generator in $\mathsf{Pos}_T$. Since $\mathsf{Pos}_T(P,-)$ preserves epimorphisms, $P_T$ is projective. Because $\mathsf{Pos}_T(P,-)(P) = \mathsf{Pos}_T(P,P) \cong S_S \in \mathsf{Pos}_S$ is a cyclic right $S$-poset, it is indecomposable, and hence also $P_T$ is indecomposable because $\mathsf{Pos}_T(P,-)$ reflects coproducts (disjoint unions). Thus $P_T$ is an indecomposable projective generator and hence a cyclic projective generator. By Lemma 2, ${}_SP_T$ is faithfully balanced. By Proposition 4, ${}_SP$ is a cyclic projective. By the dual of Proposition 4, ${}_SP$ is a generator.

2. $\Rightarrow$ 3. Assume that ${}_SP_T$ is faithfully balanced and ${}_SP, P_T$ are cyclic projective generators. Then $T \cong {}_S\mathsf{Pos}(P,P)$ as pomonoids, but due to Lemma 5 also as $(T,T)$-biposets, and similarly $S \cong \mathsf{Pos}_T(P,P)$ in ${}_S\mathsf{Pos}_S$. Hence, for the biposet ${}_TP^*_S := \mathsf{Pos}_T(P,T) \in {}_T\mathsf{Pos}_S$ we have isomorphisms

$$P^* \otimes_S P = \mathsf{Pos}_T(P,T) \otimes_S P \underset{\text{Proposition 6}}{\cong} \mathsf{Pos}_T({}_S\mathsf{Pos}(P,P),T) \underset{\text{faithfully balanced}}{\cong} \mathsf{Pos}_T(T,T) \underset{\text{Lemma 3}}{\cong} T$$

in ${}_T\mathsf{Pos}_T$, and

$$P \otimes_T P^* = P \otimes_T \mathsf{Pos}_T(P,T) \underset{\text{Proposition 5}}{\cong} \mathsf{Pos}_T(P,P) \underset{\text{faithfully balanced}}{\cong} S$$

in ${}_S\mathsf{Pos}_S$.

3. $\Rightarrow$ 1. For $P, P^*$ consider the Pos-functors $F = - \otimes_S P : \mathsf{Pos}_S \to \mathsf{Pos}_T$ and $G = - \otimes_T P^* : \mathsf{Pos}_T \to \mathsf{Pos}_S$. For every $A_S \in \mathsf{Pos}_S$,

$$(GF)(A_S) = (A \otimes_S P) \otimes_T P^* \cong A \otimes_S (P \otimes_T P^*) \cong A \otimes_S S \cong A$$

in $\mathsf{Pos}_S$ and all these isomorphisms are natural in $A$. Hence $GF \cong 1_{\mathsf{Pos}_S}$, and similarly $FG \cong 1_{\mathsf{Pos}_T}$. Since $G$ is a Pos-equivalence, by Theorem 2(a) there exists a biposet ${}_SQ^*_T \in {}_S\mathsf{Pos}_T$ such that $G \cong \mathsf{Pos}_T(Q^*,-)$. By part (d) of the same theorem, $- \otimes_S Q^*$ is an inverse of $- \otimes_T P^* = G$. Since also $F$ is an inverse of $G$, $- \otimes_S Q^* \cong F = - \otimes_S P$. By Lemma 4, $Q^* \cong P$ in ${}_S\mathsf{Pos}_T$. Since $G$ is a Pos-equivalence and $G \cong \mathsf{Pos}_T(Q^*,-)$, ${}_SQ^*_T$ is Pos-prodense, and, by Lemma 6 so is ${}_SP_T$. $\square$

From Corollary 1 and Theorem 5 we obtain the following result.

**Corollary 3.** *Pomonoids $S$ and $T$ are Morita equivalent if and only if there exist biposets ${}_SP_T \in {}_S\mathsf{Pos}_T$ and ${}_TQ_S \in {}_T\mathsf{Pos}_S$ such that*

$$\begin{aligned} P\otimes_T Q &\cong S \qquad \text{in } {}_S\mathsf{Pos}_S,\\ Q\otimes_S P &\cong T \qquad \text{in } {}_T\mathsf{Pos}_T. \end{aligned}$$

## 4. MORITA CONTEXTS

In this section we consider Morita contexts for pomonoids and prove Morita I.

**Definition 4.** *A **Morita context** is a six-tuple $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$, where $S$ and $T$ are pomonoids, ${}_SP_T \in {}_S\mathsf{Pos}_T$, ${}_TQ_S \in {}_T\mathsf{Pos}_S$, and*

$$\theta : {}_S(P\otimes_T Q)_S \to {}_SS_S, \quad \phi : {}_T(Q\otimes_S P)_T \to {}_TT_T$$

*are biposet morphisms such that, for every $p,p' \in P$ and $q,q' \in Q$,*

$$\theta(p\otimes q)\cdot p' = p\cdot\phi(q\otimes p'), \quad q\cdot\theta(p\otimes q') = \phi(q\otimes p)\cdot q'.$$

**Proposition 7.** *If $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ is a Morita context, then*
1. *the mapping*
$$\hat{\phi} : P \to {}_T\mathsf{Pos}(Q,T), \quad p \mapsto \phi(-\otimes p),$$
*is a morphism in ${}_S\mathsf{Pos}$ and the mapping*
$$\overline{\phi} : Q \to \mathsf{Pos}_T(P,T), \quad q \mapsto \phi(q\otimes -),$$
*is a morphism in ${}_T\mathsf{Pos}$;*
2. *if $\theta$ is surjective, then*
   (a) *$\theta$ is an isomorphism,*
   (b) *$P_T$ and ${}_TQ$ are cyclic projectives,*
   (c) *${}_SP$ and $Q_S$ are generators,*
   (d) *$\hat{\phi}$ and $\overline{\phi}$ are isomorphisms,*
   (e) *$\lambda : S \to \mathsf{End}(P_T)$ and $\rho : S \to \mathsf{End}({}_TQ)$ are pomonoid isomorphisms.*

*Proof.* Note that the left $S$-action on ${}_T\mathsf{Pos}(Q,T)$ is defined by $(s\cdot f)(q) = f(q\cdot s)$ and the left $T$-action on $\mathsf{Pos}_T(P,T)$ is defined by $(t\cdot g)(p) = tg(p)$ (see Lemma 3).

1. The mappings $\hat{\phi}(p) = \phi(-\otimes p) = \phi\circ(-\otimes p) : Q \to T$ and $\overline{\phi}(q) = \phi(q\otimes -) = \phi\circ(q\otimes -) : P \to T$ are morphisms in ${}_T\mathsf{Pos}$ and $\mathsf{Pos}_T$, respectively, because the mapping $-\otimes p : Q \to Q\otimes P$ is a morphism in ${}_T\mathsf{Pos}$, $q\otimes - : P \to Q\otimes P$ is a morphism in $\mathsf{Pos}_T$, and $\phi$ is a morphism in ${}_T\mathsf{Pos}_T$. If $p \leqslant p'$, $p,p' \in P$, then $-\otimes p \leqslant -\otimes p'$, and hence $\hat{\phi}(p) \leqslant \hat{\phi}(p')$, which means that $\hat{\phi}$ is order preserving. Analogously $\overline{\phi}$ is order preserving. For every $s \in S$, $t \in T$, $p \in P$, and $q \in Q$,

$$\begin{aligned} \hat{\phi}(s\cdot p)(q) &= \phi(q\otimes s\cdot p) = \phi(q\cdot s\otimes p) = \phi(-\otimes p)(q\cdot s) = (s\cdot\hat{\phi}(p))(q),\\ \overline{\phi}(t\cdot q)(p) &= \phi(t\cdot q\otimes p) = t\phi(q\otimes p) = t(\overline{\phi}(q)(p)) = (t\cdot\overline{\phi}(q))(p). \end{aligned}$$

Thus $\hat{\phi}$ is a morphism in ${}_S\mathsf{Pos}$ and $\overline{\phi}$ in ${}_T\mathsf{Pos}$.

2. Assume that $\theta$ is surjective and let $1 = \theta(p_1\otimes q_1)$, where $p_1 \in P$, $q_1 \in Q$.

(a) We need to prove that $\theta$ reflects order. Indeed, if $\theta(p\otimes q) \leqslant \theta(p'\otimes q')$, then

$$\begin{aligned} p\otimes q &= \theta(p_1\otimes q_1)\cdot p\otimes q = p_1\cdot\phi(q_1\otimes p)\otimes q = p_1\otimes\phi(q_1\otimes p)\cdot q\\ &= p_1\otimes q_1\cdot\theta(p\otimes q) \leqslant p_1\otimes q_1\cdot\theta(p'\otimes q') = p_1\otimes\phi(q_1\otimes p')\cdot q'\\ &= p_1\cdot\phi(q_1\otimes p')\otimes q' = \theta(p_1\otimes q_1)\cdot p'\otimes q' = p'\otimes q'. \end{aligned}$$

(b) For the morphisms $l_{p_1} : T \to P$ and $\phi(q_1 \otimes -) : P \to T$ in $\mathsf{Pos}_T$ we have

$$(l_{p_1} \circ \phi(q_1 \otimes -))\,(p) = p_1 \cdot \phi(q_1 \otimes p) = \theta(p_1 \otimes q_1) \cdot p = p$$

for every $p \in P$. Thus $P_T$ is a retract of $T_T$, that is, a cyclic projective by Proposition 1. For ${}_TQ$ the proof is analogous.

(c) For every $s = \theta(p \otimes q) \in S$ we can calculate

$$\begin{aligned} s &= \theta(p \otimes q)\theta(p_1 \otimes q_1) = \theta(p \otimes q \cdot \theta(p_1 \otimes q_1)) = \theta(p \otimes \phi(q \otimes p_1) \cdot q_1) \\ &= \theta(p \cdot \phi(q \otimes p_1) \otimes q_1) = \theta(- \otimes q_1)(p \cdot \phi(q \otimes p_1)), \end{aligned}$$

and hence the left $S$-poset homomorphism $\theta(- \otimes q_1) : {}_SP \to {}_SS$ is an epimorphism. Consequently, ${}_SP$ (and, symmetrically, $Q_S$) is a generator by Theorem 1.

(d) We define a mapping $\hat{\psi} : {}_T\mathsf{Pos}(Q,T) \to P$ by

$$\hat{\psi}(h) := p_1 \cdot h(q_1).$$

Obviously, $\hat{\psi}$ is order preserving. Note that, for every $h \in {}_T\mathsf{Pos}(Q,T)$ and $q \in Q$,

$$h(q) = h(q \cdot 1) = h(q \cdot \theta(p_1 \otimes q_1)) = h(\phi(q \otimes p_1) \cdot q_1) = \phi(q \otimes p_1)h(q_1).$$

Therefore,

$$\begin{aligned} \hat{\psi}(s \cdot h) &= p_1((s \cdot h)(q_1)) = p_1 \cdot h(q_1 \cdot s) = p_1 \cdot \phi(q_1 \cdot s \otimes p_1)h(q_1) \\ &= (p_1 \cdot \phi(q_1 \cdot s \otimes p_1)) \cdot h(q_1) = (\theta(p_1 \otimes q_1)s \cdot p_1) \cdot h(q_1) \\ &= s \cdot (p_1 \cdot h(q_1)) = s \cdot \hat{\psi}(h) \end{aligned}$$

for every $s \in S$, i.e. $\hat{\psi}$ is a morphism in ${}_S\mathsf{Pos}$. Moreover, the equalities

$$\begin{aligned} (\hat{\phi}\hat{\psi})(h)(q) &= \hat{\phi}(p_1 \cdot h(q_1))(q) = \phi(q \otimes p_1 \cdot h(q_1)) = \phi(q \otimes p_1)h(q_1) = h(q), \\ (\hat{\psi}\hat{\phi})(p) &= \hat{\psi}(\phi(- \otimes p)) = p_1 \cdot \phi(q_1 \otimes p) = \theta(p_1 \otimes q_1) \cdot p = p, \end{aligned}$$

$p \in P$, $q \in Q$, $h \in {}_T\mathsf{Pos}(Q,T)$, prove that $\hat{\phi}$ and $\hat{\psi}$ are mutually inverse isomorphisms in ${}_S\mathsf{Pos}$. The inverse $\overline{\psi} : \mathsf{Pos}_T(P,T) \to Q$ of $\overline{\phi}$ is defined by $\overline{\psi}(g) = g(p_1) \cdot q_1$.

(e) By Proposition 3, the mapping $\lambda : s \mapsto \lambda_s : P_T \to P_T$ is a pomonoid homomorphism. We define a mapping $\mu : \mathsf{End}(P_T) \to S$ by

$$\mu(h) := \theta(h(p_1) \otimes q_1).$$

Then $\mu(1_P) = 1$ and

$$\begin{aligned} \mu(h_1)\mu(h_2) &= \theta(h_1(p_1) \otimes q_1)\theta(h_2(p_1) \otimes q_1) = \theta(h_1(p_1) \otimes q_1 \cdot \theta(h_2(p_1) \otimes q_1)) \\ &= \theta(h_1(p_1) \otimes \phi(q_1 \otimes h_2(p_1)) \cdot q_1) = \theta(h_1(p_1) \cdot \phi(q_1 \otimes h_2(p_1)) \otimes q_1) \\ &= \theta(h_1(p_1 \cdot \phi(q_1 \otimes h_2(p_1))) \otimes q_1) = \theta(h_1(\theta(p_1 \otimes q_1) \cdot h_2(p_1)) \otimes q_1) \\ &= \theta(h_1(h_2(p_1)) \otimes q_1) = \mu(h_1 \circ h_2) \end{aligned}$$

for every $h_1, h_2 \in \mathsf{End}(P_T)$. Also $\mu$ is order preserving, and hence a homomorphism of pomonoids. Finally,

$$\begin{aligned} (\mu\lambda)(s) &= \mu(\lambda_s) = \theta(s \cdot p_1 \otimes q_1) = s\theta(p_1 \otimes q_1) = s, \\ (\lambda\mu)(h)(p) &= \lambda_{\theta(h(p_1) \otimes q_1)}(p) = \theta(h(p_1) \otimes q_1) \cdot p = h(p_1) \cdot \phi(q_1 \otimes p) \\ &= h(p_1 \cdot \phi(q_1 \otimes p)) = h(\theta(p_1 \otimes q_1) \cdot p) = h(p) \end{aligned}$$

for every $s \in S$, $p \in P$ and $h \in \mathsf{End}(P_T)$, so $\lambda$ and $\mu$ are isomorphisms. The proof for $\rho$ is analogous. □

**Proposition 8.** *If $P_T \in \mathsf{Pos}_T$ and $S = \mathsf{End}(P_T)$, then ${}_SP_T \in {}_S\mathsf{Pos}_T$. If $P^* = \mathsf{Pos}_T(P,T)$, then ${}_TP^*_S \in {}_T\mathsf{Pos}_S$. Moreover,*
1. *there is a Morita context $(S,T,{}_SP_T,{}_TP^*_S,\theta_P,\phi_P)$, where*

$$\theta_P(p\otimes q) = p\cdot q(-) = l_p \circ q \quad \text{and} \quad \phi_P(q\otimes p) = q(p),$$

2. *$P_T$ is a cyclic projective if and only if $\theta_P$ is surjective,*
3. *$P_T$ is a generator if and only if $\phi_P$ is surjective,*
4. *$P_T$ is a cyclic projective generator if and only if $\theta_P$ and $\phi_P$ are both surjective.*

*Proof.* The required actions are defined by

$$\begin{aligned} s\cdot p &:= s(p),\\ (t\cdot q)(p) &:= tq(p),\\ (q\cdot s)(p) &:= q(s\cdot p) = q(s(p)), \end{aligned}$$

$s \in \mathsf{End}(P_T)$, $p\in P$, $t\in T$, $q\in P^*$ (see Lemma 3 for the actions on $P^*$).

1. For every $p \in P$ and $q \in \mathsf{Pos}_T(P,T)$, the composite $l_p \circ q$ of two morphisms in $\mathsf{Pos}_T$ is an endomorphism of $P_T$. Note that

$$(l_{p\cdot t}\circ q)(p') = (p\cdot t)\cdot q(p') = p\cdot (tq(p')) = l_p((t\cdot q)(p')) = (l_p \circ (t\cdot q))(p')$$

for every $p,p' \in P$, $t\in T$, $q \in P^*$, so $l_{p\cdot t}\circ q = l_p\circ(t\cdot q)$. If now $p\otimes q \leqslant p'\otimes q'$ in $P\otimes_T P^*$, i.e.

$$\begin{array}{rclcrcl} p &\leqslant& p_1\cdot t_1 & & & & \\ p_1\cdot u_1 &\leqslant& p_2\cdot t_2 & & t_1\cdot q &\leqslant& u_1\cdot q_2\\ p_2\cdot u_2 &\leqslant& p_3\cdot t_3 & & t_2\cdot q_2 &\leqslant& u_2\cdot q_3\\ &\dots& & & &\dots& \\ p_n\cdot u_n &\leqslant& p' & & t_n\cdot q_n &\leqslant& u_n\cdot q', \end{array}$$

for some $p_1,\dots,p_n\in P$, $q_2,\dots,q_n\in P^*$ and $t_1,\dots,t_n,u_1,\dots,u_n \in T$, then

$$l_p\circ q \leqslant l_{p_1\cdot t_1}\circ q = l_{p_1}\circ(t_1\cdot q) \leqslant l_{p_1}\circ(u_1\cdot q_2) = l_{p_1\cdot u_1}\circ q_2 \leqslant \dots \leqslant l_{p_n\cdot u_n}\circ q' \leqslant l_{p'}\circ q'.$$

Hence $\theta_P$ is well defined and order preserving.

If $q\otimes p \leqslant q'\otimes p'$ in $P^*\otimes_S P$, $p,p'\in P$, $q,q'\in P^*$, then we have inequalities

$$\begin{array}{rclcrcl} q &\leqslant& q_1\cdot s_1 & & & & \\ q_1\cdot z_1 &\leqslant& q_2\cdot s_2 & & s_1\cdot p &\leqslant& z_1\cdot p_2\\ q_2\cdot z_2 &\leqslant& q_3\cdot s_3 & & s_2\cdot p_2 &\leqslant& z_2\cdot p_3\\ &\dots& & & &\dots& \\ q_n\cdot z_n &\leqslant& q' & & s_n\cdot p_n &\leqslant& z_n\cdot p' \end{array}$$

for some $p_2,\dots,p_n\in P$, $q_1,\dots,q_n \in P^*$, $s_1,\dots,s_n,z_1,\dots,z_n\in S$, and hence

$$\begin{aligned} q(p) &\leqslant (q_1\cdot s_1)(p) = q_1(s_1\cdot p) \leqslant q_1(z_1\cdot p_2) = (q_1\cdot z_1)(p_2) \leqslant (q_2\cdot s_2)(p_2)\\ &\leqslant \dots \leqslant (q_n\cdot s_n)(p_n) = q_n(s_n\cdot p_n) \leqslant q_n(z_n\cdot p') = (q_n\cdot z_n)(p') \leqslant q'(p'). \end{aligned}$$

Therefore $\phi_P$ is well defined and order preserving. From

$$\begin{aligned}
\theta_P(s\cdot p\otimes q\cdot s')(p') &= (s\cdot p)\cdot((q\cdot s')(p')) = (s\cdot p)\cdot(q(s'(p')))\\
&= s\cdot(p\cdot q(s'(p'))) = s(\theta_P(p\otimes q)(s'(p')))\\
&= (s\circ\theta_P(p\otimes q)\circ s')(p'),\\
\phi_P(t\cdot q\otimes p\cdot t') &= (t\cdot q)(p\cdot t') = tq(p\cdot t') = tq(p)t' = t\phi_P(q\otimes p)t',
\end{aligned}$$

$s,s'\in S$, $t,t'\in T$, $p,p'\in P$, $q\in P^*$, it follows that $\theta_P$ and $\phi_P$ are biposet morphisms. Also,

$$\begin{aligned}
\theta_P(p\otimes q)\cdot p' &= (p\cdot q(-))\cdot p' = (p\cdot q(-))(p') = p\cdot q(p')\\
&= p\cdot\phi_P(q\otimes p'),\\
(q\cdot\theta_P(p\otimes q'))(p') &= q((p\cdot q'(-))(p')) = q(p\cdot q'(p')) = q(p)q'(p')\\
&= \phi_P(q\otimes p)q'(p') = (\phi_P(q\otimes p)\cdot q')(p'),
\end{aligned}$$

for all $p,p'\in P$, $q,q'\in P^*$.

2. *Necessity*. If $P_T$ is a cyclic projective, then by Proposition 1 there exist morphisms $P\underset{\beta}{\overset{\alpha}{\rightleftarrows}}T$ in $\mathsf{Pos}_T$ such that $\beta\circ\alpha = 1_P$. Take any $f\in\mathsf{End}(P_T)$. Since

$$(l_{\beta(1)}\circ(\alpha\circ f))(p) = \beta(1)\cdot\alpha(f(p)) = \beta(\alpha(f(p))) = f(p)$$

for every $p\in P$, we have $\theta_P(\beta(1)\otimes(\alpha\circ f)) = l_{\beta(1)}\circ(\alpha\circ f) = f$.

*Sufficiency*. If $\theta_P$ is surjective then there exist $p\in P$ and $q\in\mathsf{Pos}_T(P,T)$ such that $1_P = \theta_P(p\otimes q) = l_p\circ q$. Thus $P_T$ is a cyclic projective by Proposition 1.

3. *Necessity*. If $P_T$ is a generator, then, by Theorem 1, there exist morphisms $P\underset{\delta}{\overset{\gamma}{\rightleftarrows}}T$ in $\mathsf{Pos}_T$ such that $\gamma\circ\delta = 1_T$. Hence, for every $t\in T$, $\phi_P(\gamma\otimes\delta(t)) = \gamma(\delta(t)) = t$.

*Sufficiency*. If $\phi_P$ is surjective, then there exist $p\in P$, $q\in\mathsf{Pos}_T(P,T)$ such that $1 = \phi_P(q\otimes p) = q(p)$. Hence $(q\circ l_p)(t) = q(p\cdot t) = q(p)\cdot t = t$ for every $t\in T$, that is, $q\circ l_p = 1_T$. By Theorem 1, $P_T$ is a generator.

4. This follows from 2 and 3. □

**Theorem 6.** *Pomonoids $S$ and $T$ are Morita equivalent if and only if there exists a Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ with $\theta$ and $\phi$ surjective.*

*Proof. Necessity* follows from Proposition 8 and Theorem 3.

*Sufficiency*. Suppose that $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ is a Morita context with $\theta$ and $\phi$ surjective. Then, by Proposition 7, $P_T$ is a cyclic projective. By the analogue of Proposition 7, $P_T$ is a generator and $\lambda: T\to\mathsf{End}(Q_S)$ is a pomonoid isomorphism. Hence $S$ and $T$ are Morita equivalent by Theorem 3. □

**Theorem 7** (Morita I)**.** *Let $P_T$ be a cyclic projective generator*, $S = \mathsf{End}(P_T)$ *and* ${}_TQ_S = \mathsf{Pos}_T(P,T)$. *Then*
1. $-\otimes_T Q : \mathsf{Pos}_T\to\mathsf{Pos}_S$ *and* $-\otimes_S P : \mathsf{Pos}_S\to\mathsf{Pos}_T$ *are mutually inverse* $\mathsf{Pos}$*-equivalence functors*;
2. $P\otimes_T - : {}_T\mathsf{Pos}\to{}_S\mathsf{Pos}$ *and* $Q\otimes_S - : {}_S\mathsf{Pos}\to{}_T\mathsf{Pos}$ *are mutually inverse* $\mathsf{Pos}$*-equivalence functors.*

*Proof.* 1. From Proposition 8 it follows that there exists a Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta_P,\phi_P)$ with $\theta_P$ and $\phi_P$ surjective. By Proposition 7, ${}_SP$ is a generator, $P_T$ is a cyclic projective, and $\lambda: S\to\mathsf{End}(P_T)$ is a pomonoid isomorphism. By the analogue of Proposition 7, ${}_SP$ is a cyclic projective, $P_T$ is a generator, and $\rho: T\to\mathsf{End}({}_SP)$ is a pomonoid isomorphism. Hence ${}_SP_T$ is $\mathsf{Pos}$-prodense by Theorem 5, and the functor $G = \mathsf{Pos}_T(P,-) : \mathsf{Pos}_T\to\mathsf{Pos}_S$ is a $\mathsf{Pos}$-equivalence. By Theorem 2(b) there exists a biposet ${}_TQ'_S$ such that $G\cong -\otimes_T Q'$, but then ${}_TQ_S = G(T)\cong{}_TQ'_S$ and $G\cong -\otimes_T Q$. By Theorem 2(d), $-\otimes_S P : \mathsf{Pos}_S\to\mathsf{Pos}_T$ is a $\mathsf{Pos}$-equivalence inverse to $-\otimes_T Q$.

2. This can be proven similarly. □

## 5. PICARD GROUPS

In this section we give a proof of Morita III for pomonoids.

Consider the category $\mathscr{P}$, where
- objects are pomonoids,
- morphisms $T \longrightarrow S$ are isomorphism classes $[P]$ of Pos-prodense biposets ${}_SP_T \in {}_S\mathsf{Pos}_T$,
- the composite of $T \xrightarrow{[P]} S \xrightarrow{[X]} U$ is defined by

$$[X] \circ [P] := [{}_U(X \otimes_S P)_T],$$

- the identity morphism of a pomonoid $S$ is the isomorphism class $[S]$ of the Pos-prodense biposet ${}_SS_S$.

To see that the composition is well defined, suppose that $P \cong P'$ in ${}_S\mathsf{Pos}_T$ and $X \cong X'$ in ${}_U\mathsf{Pos}_S$. Then, since the functors $X \otimes_S - : {}_S\mathsf{Pos}_T \to {}_U\mathsf{Pos}_T$ and $- \otimes_S P' : {}_U\mathsf{Pos}_S \to {}_U\mathsf{Pos}_T$ preserve isomorphisms, $X \otimes P \cong X \otimes P' \cong X' \otimes P'$ in ${}_U\mathsf{Pos}_T$. The fact that $[S]$ is the identity morphism of an object $S$ of $\mathscr{P}$ follows from Lemma 1 and its dual. The composition is associative because the tensor multiplication of biposets is.

Let Pre be the category of preordered sets with preorder preserving mappings as morphisms.

**Proposition 9.** *The category $\mathscr{P}$ is a* Pre*-groupoid.*

*Proof.* By Theorem 5, $\mathscr{P}$ is a groupoid, where the inverse of a morphism $[P] : T \to S$ is $[P^*] : S \to T$. We write ${}_SP_T \trianglelefteq {}_SP'_T$ if there exists a regular monomorphism ${}_SP_T \to {}_SP'_T$ in ${}_S\mathsf{Pos}_T$, and we define a relation $\le$ on a mor-set $\mathscr{P}(S,T)$ by

$$[P] \le [P'] \Longleftrightarrow {}_SP_T \trianglelefteq {}_SP'_T.$$

Clearly this relation is well defined, reflexive, and transitive. Consider morphisms $S \underset{[P']}{\overset{[P]}{\rightrightarrows}} T \xrightarrow{[Q]} U$ in $\mathscr{P}$ such that $[P] \le [P']$. Since ${}_UQ_T$ is Pos-prodense, $Q_T$ is projective and hence po-flat in $\mathsf{Pos}_T$ (Theorem 3.23 of [11]). This means that the functor $Q_T \otimes - : {}_T\mathsf{Pos} \to \mathsf{Pos}$ preserves regular monomorphisms, but then also the functor ${}_UQ_T \otimes - : {}_T\mathsf{Pos}_S \to {}_U\mathsf{Pos}$ preserves regular monomorphisms, in particular ${}_U(Q \otimes_T P)_S \trianglelefteq {}_U(Q \otimes_T P')_S$. Consequently,

$$[Q] \circ [P] = [Q \otimes P] \le [Q \otimes P'] = [Q] \circ [P']$$

and the preorder $\le$ is compatible with the composition from the left. Similarly it is compatible with the composition from the right and therefore $\mathscr{P}$ is a Pre-category. □

**Corollary 4.** *The endomorphism monoid $\mathscr{P}(S,S)$ of a pomonoid $S$ in $\mathscr{P}$ is a group.*

**Definition 5.** *We denote the group $\mathscr{P}(S,S)$ by* $\mathsf{Pic}(S)$ *and call it the* **Picard group** *of a pomonoid $S$.*

**Corollary 5.** *Picard groups of Morita equivalent pomonoids are isomorphic.*

*Proof.* Due to Corollary 1, two pomonoids are Morita equivalent if and only if they are isomorphic objects in $\mathscr{P}$. Endomorphism monoids of isomorphic objects of a category are isomorphic. □

Consider the category $\mathscr{M}$, where
- objects are the categories $\mathsf{Pos}_S$, where $S$ is a pomonoid,
- morphisms $\mathsf{Pos}_S \longrightarrow \mathsf{Pos}_T$ are isomorphism classes $[F]$ of Pos-equivalence functors $F : \mathsf{Pos}_S \longrightarrow \mathsf{Pos}_T$,
- the composition is given by the composition of functors.

**Theorem 8** (Morita III)**.** *The categories $\mathscr{M}$ and $\mathscr{P}$ are dually isomorphic.*

*Proof.* We define contravariant functors $\mathscr{M} \underset{L}{\overset{K}{\rightleftarrows}} \mathscr{P}$ by the assignments

$$\begin{tikzcd} \mathsf{Pos}_S \arrow[r, mapsto] \arrow[d, "{[F]}"'] & S \\ \mathsf{Pos}_T \arrow[r, mapsto] & T \arrow[u, "{[_SF(S)_T]}"'] \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, mapsto] & \mathsf{Pos}_S \arrow[d, "{[-\otimes_S P]}"] \\ T \arrow[u, "{[_SP_T]}"] \arrow[r, mapsto] & \mathsf{Pos}_T \end{tikzcd},$$

respectively. Let $F : \mathsf{Pos}_S \to \mathsf{Pos}_T$ and $G : \mathsf{Pos}_T \to \mathsf{Pos}_U$ be Pos-equivalence functors. By Theorem 2, $G \cong - \otimes_T G(T)$, so ${}_SG(F(S))_U \cong {}_S(F(S) \otimes_T G(T))_U$ and

$$\begin{aligned} K([G]\circ[F]) &= K([G\circ F]) = [{}_SG(F(S))_U] = [{}_S(F(S)\otimes_T G(T))_U] \\ &= [{}_SF(S)_T]\circ[{}_TG(T)_U] = K([F])\circ K([G]). \end{aligned}$$

If ${}_SP_T \in {}_S\mathsf{Pos}_T$ and ${}_UX_S \in {}_U\mathsf{Pos}_S$, then

$$\begin{aligned} L([{}_UX_S]\circ[{}_SP_T]) &= L([{}_U(X\otimes_S P)_T]) = [-\otimes_U (X\otimes_S P)] \\ &= [(-\otimes_S P)\circ(-\otimes_U X)] = [-\otimes_S P]\circ[-\otimes_U X] \\ &= L([{}_SP_T])\circ L([{}_UX_S]). \end{aligned}$$

It is easy to see that $K$ and $L$ preserve identities. Moreover, by Lemma 1 and Theorem 2,

$$\begin{aligned} (KL)([{}_SP_T]) &= K([-\otimes_S P]) = [{}_S(S\otimes P)_T] = [{}_SP_T], \\ (LK)([F]) &= L([{}_SF(S)_T]) = [-\otimes_S F(S)] = [F]. \end{aligned}$$

□

**Remark 2.** Let us write $F \trianglelefteq F'$ if there is a regular monomorphism $\mu : F \to F'$ in the category of Pos-functors from $\mathsf{Pos}_S$ to $\mathsf{Pos}_T$, and define

$$[F] \le [F'] \Longleftrightarrow F \trianglelefteq F'.$$

This way $\mathscr{M}$ becomes a Pre-category.

By Theorem 3, two pomonoids $S$ and $T$ are Morita equivalent if and only if the full subcategories $\mathsf{CPG}_S$ and $\mathsf{CPG}_T$ of $\mathsf{Pos}_S$ and $\mathsf{Pos}_T$ generated by the cyclic projective generators are Pos-equivalent. Let us consider the Pre-category $\overline{\mathscr{M}}$, where objects are categories $\mathsf{CPG}_S$ ($S$ is a pomonoid), morphisms of $\overline{\mathscr{M}}$ are the isomorphism classes of Pos-equivalence functors between them, and a preorder of morphisms is defined as above. Then the functors

$$\overline{\mathscr{M}} \underset{\overline{L}}{\overset{\overline{K}}{\rightleftarrows}} \mathscr{P}$$

that are defined similarly to $K$ and $L$ are mutually inverse isomorphisms. Moreover, $\overline{K}$ and $\overline{L}$ are Pre-functors (and hence $\overline{\mathscr{M}}$ and $\mathscr{P}$ are isomorphic as Pre-categories). Indeed, if $[F] \le [F']$, then $\overline{K}([F]) = [{}_SF(S)_T] \leqslant [{}_SF'(S)_T] = \overline{K}([F'])$. If $[{}_SP_T] \le [{}_SP'_T]$, then there is a regular monomorphism $m : P \to P'$ in ${}_S\mathsf{Pos}_T$. Therefore $L([{}_SP_T]) = [-\otimes_S P] \leqslant [-\otimes_S P'] = L([{}_SP'_T])$, where $-\otimes_S P \trianglelefteq -\otimes_S P'$ because $\mu = (\mu_A)_{A\in\mathsf{CPG}_S} : -\otimes_S P \to -\otimes_S P'$ with

$$\mu_A = 1_A \otimes m : (A\otimes_S P)_T \longrightarrow (A\otimes_S P')_T, \quad a\otimes p \mapsto a\otimes m(p'),$$

is a regular monomorphism in $\mathsf{Pos}_T$, because the cyclic projective generator $A_S$ is po-flat.

**ACKNOWLEDGEMENTS**

This research was supported by the Estonian Science Foundation under grant No. 8394 and Estonian Targeted Financing Project SF0180039s08. I am grateful to the referees for useful suggestions.

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## Morita-teoreemid osaliselt järjestatud monoidide jaoks

Valdis Laan

Osaliselt järjestatud monoide $S$ ja $T$ nimetatakse Morita-ekvivalentseteks, kui parempoolsete järjestatud $S$-polügoonide kategooria ning parempoolsete järjestatud $T$-polügoonide kategooria on ekvivalentsed kui üle osaliselt järjestatud hulkade kategooria Pos rikastatud kategooriad. Me anname Pos-protihedate bipolügoonide (üle osaliselt järjestatud monoidide) kirjelduse ja tõestame Morita-teoreemid I, II ja III.