



The concepts of Lie derivative for discrete-time systems

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Abstract. The paper extends the concept of the Lie derivative of the vector field, used in the study of the continuous-time dynamical systems, for the discrete-time case. In the continuous-time case the Lie derivative of a vector field (1-form or scalar function) with respect to the system dynamics is defined as its rate of change in time. In the discrete-time case we introduce the algebraic definition of the Lie derivative, using the concepts of forward and backward shifts. The definitions of discrete-time forward and backward shifts of the vector field are based on the concepts of already known forward and backward shifts of the 1-forms and on the scalar product of 1-form and vector field. Further we show that the interpretation of the discrete-time Lie derivative agrees with its interpretation as the rate of change in the continuous-time case. Finally, the geometric property of the discrete-time Lie derivative is also examined and shown to mimic the respective property in the continuous-time case.

Key words: differential geometry, difference equations, vector fields, Lie derivative, forward shift, backward shift.

1. INTRODUCTION

The Lie derivative is an important concept of differential geometry (see e.g. [1]). In continuous time the vector field defining the dynamics of the system generates a one-parameter group of transformations, called the *flow*. The Lie derivative of a function, vector field or 1-form with respect to the generating vector field is defined as its rate of change along the trajectories of the generating vector field.

The aim of this paper is to extend the concept of the Lie derivative of the vector field to the discrete-time case, with a long-range goal to work out a more general theory covering both the continuous- and discrete-time systems using the tools of time scale calculus [2]. In doing so one has to assume that the state transition map of the discrete-time system is reversible, the property not necessarily satisfied automatically in the discrete-time case, except for models obtained by sampling the continuous-time systems [3]. Some concepts related to the discrete-time Lie derivative of a vector field, defined in this paper, have already been used in some earlier works [4–8], although the concept itself has never been introduced. For example, in [6] the forward shift of the vector field was defined and used in the study of the accessibility. Moreover, in [4] and [5] the so-called Ad-operator was introduced and used in addressing the problems of feedback linearization and transformation of equations into the observer form, respectively. The Ad-operator may be interpreted as the backward shift of the vector field, though this interpretation was not given in [4,5]. In [7] the it-operator¹ was introduced in the analogy of the Ad-operator in the continuous-time case,

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¹ “It” stands for iterated.

which again may be interpreted as the backward shift. Rieger, Schlacher, and Holl in [8] are close to the introduction of the concept of the discrete-time Lie derivative. Namely, in [8] the specific geometric property of a Lie derivative of a vector field has been extended into the discrete-time domain. According to this property, the Lie derivative of a vector field is identically zero if and only if the one-parameter group of transformations generated by the vector field commutes with the flow. However, the new concept was not explicitly introduced but was hidden in the solution of the specific (observability analysis) problem. Note that papers [4,6–8] study the dynamical systems with the inputs, whereas [5] addresses the input-free case. Since the concept of the Lie derivative plays a key role in the solutions of numerous (control) problems, it is advisable explicitly to extend the concept to the discrete-time case, and study its properties. In this paper, in order to simplify the presentation, we focus on the input-free case.

Finally, note that the concept of the discrete Lie derivative was introduced some time ago within the framework of discrete exterior calculus (DEC), developed with the aim to preserve, in discrete space, the structure and invariants of smooth (continuous) theory ([9,10]). Unlike DEC that addresses spatial discretization, we discretize the time.

The paper is organized as follows. In Section 2 the interpretation of the Lie derivative of a vector field with respect to the time derivative operator is recalled. Section 3 describes the discrete-time nonlinear system together with the algebraic structures associated with it, which are necessary in our studies, including the definitions of the forward and backward shift operators of the vector field. Moreover, in this section the algebraic definition of the concept of the Lie derivative is given. It is shown that this definition may be interpreted as the discrete-time analogue of the corresponding continuous-time interpretation recalled in Section 2. Section 4 focuses on the geometric property of the Lie derivative and relates the results of this paper to those of [8]. Section 5 contains the examples and Section 6 concludes the paper.

2. INTERPRETATION OF THE LIE DERIVATIVE

The Lie derivative of a vector field Ξ with respect to another vector field F , denoted by $L_F \Xi$, describes the rate of change of Ξ along a trajectory of the vector field F . Although this is a well-known fact, we will demonstrate it below step by step, in order to repeat the analogous steps in the extension of the concept of the Lie derivative to the discrete-time case. A vector field

$$F = \sum_{i=1}^n F_i(x) \frac{\partial}{\partial x_i} \quad (1)$$

defined on \mathbb{R}^n generates a flow $\Phi_\tau = e^{\tau F}$, given by the solution of the differential equation

$$\dot{x} = F(x). \quad (2)$$

The flow Φ_τ maps a point $x(t)$ in \mathbb{R}^n , corresponding to the time instant t , into another point $x(t + \tau)$, corresponding to the time instant $t + \tau$, according to

$$x(t + \tau) = \Phi_\tau(x(t)) = x(t) + F(x(t))\tau + \left. \frac{\partial F}{\partial x} \right|_{x=x(t)} F(x(t)) \frac{\tau^2}{2} + \dots \quad (3)$$

The flow Φ_τ is called *the exponential map*, corresponding to the vector field F ; the vector field F is called *the infinitesimal generator* of Φ_τ , or, according to (2), *the time derivative operator*. Formula (3) describes *the trajectory* of the vector field F in time, starting from the initial point $x(t)$.

Compute next the rate of change of a vector field $\Xi = \sum_{i=1}^n \Xi_i(x) \partial/\partial x_i$ along the trajectory (3) in three steps. The vector field Ξ defines two vectors,

$$\Xi(x(t)) = \sum_{i=1}^n \Xi_i(x(t)) \frac{\partial}{\partial x_i} \quad (4)$$

at the initial point of the trajectory, and

$$\Xi(x(t + \tau)) = \sum_{i=1}^n \Xi_i(x(t + \tau)) \frac{\partial}{\partial x_i} \tag{5}$$

at the endpoint of the trajectory. First, compute the components of $\Xi(x(t + \tau))$ given by (5) in the first approximation with respect to τ using the Taylor series and replacing in the latter the difference $x_i(t + \tau) - x_i(t)$ by $F_i(x(t))\tau$; see (3):

$$\Xi_i(x(t + \tau)) \approx \Xi_i(x(t)) + \sum_{j=1}^n \frac{\partial \Xi_i}{\partial x_j} \Big|_{x=x(t)} F_j(x(t))\tau. \tag{6}$$

Second, compute the change $\Delta\Xi$ of the vector field Ξ along the trajectory of F , subtracting from the vector $\Xi(x(t + \tau))$ the vector $(T\Phi_\tau \cdot \Xi)(x(t))$, where the latter may be interpreted as $\Xi(x(t))$, transferred along the trajectory of F from the point $x(t)$ into the point $x(t + \tau)$ by the tangent map $T\Phi_\tau$ of the flow Φ_τ . Computing the partial derivative of (3) with respect to x and taking the first approximation with respect to τ , one gets

$$T\Phi_\tau(x(t)) \approx I + \frac{\partial F}{\partial x} \Big|_{x=x(t)} \tau. \tag{7}$$

Consequently, the i th component of the vector field $T\Phi_\tau \cdot \Xi$ reads in the first approximation as

$$(T\Phi_\tau \cdot \Xi)_i(x(t + \tau)) \approx \Xi_i(x(t)) + \sum_{j=0}^n \frac{\partial F_i}{\partial x_j} \Big|_{x=x(t)} \Xi_j(x(t))\tau + \dots \tag{8}$$

Due to (6) and (8), the components of the vector $\Delta\Xi$, defined at the point $x(t + \tau)$, are in the first approximation as follows:

$$\begin{aligned} \Delta\Xi_i(x(t + \tau)) &= \Xi_i(x(t + \tau)) - T\Phi_\tau \cdot \Xi_i(x(t)) \\ &\approx \sum_{j=1}^n \left(\frac{\partial \Xi_i}{\partial x_j} \Big|_{x=x(t)} F_j(x(t)) - \frac{\partial F_i}{\partial x_j} \Big|_{x=x(t)} \Xi_j(x(t)) \right) \tau = [F, \Xi]_i \tau. \end{aligned}$$

The third step is to compute the rate of change by dividing $\Delta\Xi$ by τ and approaching the limit $\tau \rightarrow \infty$. The result is the Lie derivative

$$\dot{\Xi}(x(t)) = \lim_{\tau \rightarrow 0} \frac{\Delta\Xi(x(t + \tau))}{\tau} = [F, \Xi](x(t)) = (L_F \Xi)(x(t)). \tag{9}$$

Remark 1. The Lie derivative of a vector field may be and is usually defined by the alternative formula (see for example [11], p. 51)

$$(L_F \Xi)(x(t)) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} (T\Phi_\tau^{-1} \cdot \Xi(x(t + \tau)) - \Xi(x(t))). \tag{10}$$

Note that (9) and (10) can be easily shown to yield the same result if we use in (10) the first approximation of $T\Phi_\tau^{-1} = I - (\partial F / \partial x) \tau$.

3. LIE DERIVATIVE OF A VECTOR FIELD WITH RESPECT TO THE DISCRETE-TIME DYNAMICS

The goal of this section is to extend the concept of the Lie derivative of an arbitrary vector field to the discrete-time case. Consider now the discrete-time system

$$x(t+1) = \Phi(x(t)), \quad (11)$$

where $x \in X$, an open subset of \mathbb{R}^n , such that Φ is analytic diffeomorphism from X onto X . This implies that Φ is invertible (on X) and its inverse is also analytic. Denote by \mathcal{K} the inversive difference field of the meromorphic functions in variable x [6]. The *forward shift* operator σ of the variable x is defined by the state transition map Φ of the discrete-time system

$$\sigma(x) := \Phi(x), \quad (12)$$

and the *backward shift* operator $\rho := \sigma^{-1}$ of the variable x is defined by

$$\rho(x) := \Phi^{-1}(x). \quad (13)$$

Moreover, the assumption on X guarantees that X is invariant with respect to forward and backward dynamics of our system.

Hereinafter, in order to simplify the presentation, we often use the notations

$$x^\sigma = \Phi(x), \quad x^\rho = \Phi^{-1}(x). \quad (14)$$

The forward and backward shift operators $\sigma, \rho : \mathcal{K} \rightarrow \mathcal{K}$ are defined as the composite functions

$$\varphi^\sigma(x) := \varphi(\Phi(x)) \quad (15)$$

and

$$\varphi^\rho(x) := \varphi(\Phi^{-1}(x)), \quad (16)$$

respectively.

The discrete-time analogue of the Lie derivative of a function $\varphi(x)$ is its *difference*, which can be defined in two different ways, either as the *forward difference*

$$(L_\Phi^\Delta \varphi)(x) := \varphi^\Delta(x) = \varphi^\sigma(x) - \varphi(x) \quad (17)$$

(see e.g. [12], p. 270), or also as the *backward difference*

$$(L_\Phi^\nabla \varphi)(x) := \varphi^\nabla(x) = \varphi(x) - \varphi^\rho(x). \quad (18)$$

Furthermore, define the vector space $\mathcal{E} := \text{span}_{\mathcal{K}} \{dx_1, \dots, dx_n\} := \text{span}_{\mathcal{K}} \{dx\}$ [6]. Any element of \mathcal{E} , called a differential 1-form, is a vector

$$\omega(x) = \sum_{i=1}^n \omega_i(x) dx_i. \quad (19)$$

The differential operator $d : \mathcal{K} \rightarrow \mathcal{E}$ is defined as

$$d\varphi(x) := \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_i} dx_i.$$

The forward and backward shift operators $\sigma, \rho : \mathcal{E} \rightarrow \mathcal{E}$ of $\omega(x)$ in (19) are defined by

$$\omega^\sigma(x) = \sum_{i=1}^n \omega_i^\sigma(x) dx_i^\sigma = \sum_{i,j=1}^n \omega_i(\Phi(x)) \frac{\partial \Phi_i(x)}{\partial x_j} dx_j \quad (20)$$

and

$$\omega^\rho(x) = \sum_{i=1}^n \omega_i^\rho(x) dx_i^\rho = \sum_{i,j=1}^n \omega_i(\Phi^{-1}(x)) \frac{\partial (\Phi^{-1}(x))_i}{\partial x_j} dx_j, \quad (21)$$

respectively [6]. The discrete-time analogue of a Lie derivative of a 1-form is, according to [12], its forward difference

$$(L_\Phi^\Delta \omega)(x) := \omega^\sigma(x) - \omega(x). \quad (22)$$

Another possibility is to define it as the backward difference

$$(L_\Phi^\nabla \omega)(x) := \omega(x) - \omega^\rho(x). \quad (23)$$

Note that (22) mimics (17), and (23) mimics (18).

Next define the space \mathcal{E}^* , dual to \mathcal{E} , whose elements are the *vector fields*. The vector field Ξ is a map $\Xi : \mathcal{E} \rightarrow \mathcal{H}$, mapping an arbitrary 1-form $\omega \in \mathcal{E}$ into a function, called a *scalar product*

$$\Xi(\omega) = \langle \omega, \Xi \rangle \in \mathcal{H}.$$

In a *natural (canonical) basis* of \mathcal{E}^* , dual to the canonical basis $\{dx\}$ of \mathcal{E} , the vector field has the form

$$\Xi = \sum_{i=1}^n \Xi_i(x) \frac{\partial}{\partial x_i}. \quad (24)$$

We define the discrete-time forward and backward Lie derivatives of a vector field (24) analogously to formulae (22) and (23), i.e. as the forward and backward differences

$$(L_\Phi^\Delta \Xi)(x) = \Xi^\sigma(x) - \Xi(x), \quad (25)$$

$$(L_\Phi^\nabla \Xi)(x) = \Xi(x) - \Xi^\rho(x), \quad (26)$$

respectively. Definitions (25) and (26) depend on Ξ^σ and Ξ^ρ , respectively, which will be defined using the scalar products of 1-forms and vector fields, as well as formulae (20) and (21). First, define Ξ^ρ from the equality

$$\langle \omega(x), \Xi^\rho(x) \rangle = \langle \omega^\sigma(x), \Xi(x) \rangle^\rho, \quad (27)$$

taking into account that the backward shift of a scalar product as a function in \mathcal{H} is well defined. Note that, according to (15) and (20),

$$\langle \omega^\sigma(x), \Xi(x) \rangle^\rho = \langle (\omega(\Phi)) \cdot T\Phi, \Xi \rangle (\Phi^{-1}(x)),$$

where $T\Phi$ is the Jacobi matrix corresponding to the map Φ ,

$$T\Phi(x) = \left(\frac{\partial \Phi(x)}{\partial x} \right), \quad (28)$$

whose elements are the partial derivatives of Φ with respect to the states x .

Due to the associativity property of the matrix multiplication and interpreting ω as the row vector and Ξ as the column vector, one may rewrite this result as

$$\langle \omega^\sigma(x), \Xi(x) \rangle^\rho = \langle \omega(x), (T\Phi \cdot \Xi)(\Phi^{-1}(x)) \rangle. \quad (29)$$

Therefore, in case of an arbitrary 1-form, (27) holds iff

$$\Xi^\rho(x) := (T\Phi \cdot \Xi)(\Phi^{-1}(x)). \quad (30)$$

Analogously, the *forward shift* of a vector field can be defined by

$$\Xi^\sigma(x) := (T\Phi)^{-1} \cdot (\Xi(\Phi(x))). \quad (31)$$

Remark 2. Definition (30) agrees with the formulae in the books [11], p. 148, and [13], p. 145, where the authors address the question of how the vector field Ξ transforms under the coordinate transformation $z = \Phi(x)$ in \mathbb{R}^n . Interpreting x as the state at time instant $t - 1$, i.e. $x(t - 1)$ and z as $x(t)$, formula $(T\Phi \cdot \Xi)(\Phi^{-1}(z))$ in [11] and [13] corresponds to (30).

As expected, for $\sigma, \rho : \mathcal{E}^* \rightarrow \mathcal{E}^*$, $\rho\sigma = \sigma\rho = Id$. Really, applying the backward shift operator ρ , defined by (30), to Ξ^σ in (31) yields again Ξ , as does the application of the forward shift σ , defined by (31) to Ξ^ρ in (30):

$$(\Xi^\rho)^\sigma(x) = (T\Phi)^{-1}(x) \cdot (\Xi^\rho(\Phi(x))) = \left((T\Phi)^{-1} \cdot (T\Phi \cdot \Xi) \right)(x) = \Xi(x), \quad (32)$$

$$(\Xi^\sigma)^\rho(x) = (T\Phi \cdot \Xi^\sigma)(\Phi^{-1}(x)) = \Xi(\Phi(\Phi^{-1}(x))) = \Xi(x). \quad (33)$$

Remark 3. Note that only the definition of ω^σ does not require invertibility of Φ , whereas ω^ρ , Ξ^ρ , and Ξ^σ may be defined only if Φ^{-1} exists.

Next we compare the calculation of two dual operations, the forward shift of a 1-form and the backward shift of a vector field. The forward shift of a 1-form (19) may be computed in the following two steps.

1. Evaluate ω as in (19) at the point x^σ ,

$$\omega(x^\sigma) = \sum_{i=1}^n \omega_i(x^\sigma) dx_i^\sigma.$$

2. Express x^σ in terms of x using the composition with Φ , defined by system dynamics (12),

$$\omega^\sigma(x) = \sum_{i=1}^n \omega_i(\Phi(x)) d\Phi_i(x) = \sum_{i,j=1}^n \omega_i(\Phi(x)) \frac{\partial \Phi_i(x)}{\partial x_j} dx_j.$$

In case we interpret the 1-form as a row vector, this step may be rewritten as

$$\omega^\sigma(x) = \omega(\Phi(x)) \cdot T\Phi(x). \quad (34)$$

In order to calculate the backward shift of a vector field (24), we make the inverse operations in the opposite order.

1. Multiply Ξ as in (24) from the left by $T\Phi$ and express in the result² x in terms of x^σ , using the composition with $\Phi^{-1}(x^\sigma)$ defined by the inverse of system dynamics (12)

$$\Xi^\rho(x^\sigma) = (T\Phi \cdot \Xi)(\Phi^{-1}(x^\sigma)). \quad (35)$$

² Since $T\Phi \cdot \Xi$ is defined at the point x^σ .

Vector field (35), interpreted as a column vector, may be alternatively rewritten as a linear combination of the basis vectors as follows:

$$\Xi^p(x^\sigma) = \sum_{i,j=1}^n \left(\frac{\partial \Phi_i}{\partial x_j} \Xi_j \right) (\Phi^{-1}(x^\sigma)) \frac{\partial}{\partial x_i^\sigma}. \quad (36)$$

2. Evaluate (36) at the point x :

$$\Xi^p(x) = \sum_{i,j=1}^n \left(\frac{\partial \Phi_i}{\partial x_j} \Xi_j \right) (\Phi^{-1}(x)) \frac{\partial}{\partial x_i}.$$

Below we will show that the backward Lie derivative defined by (26) may be understood as the discrete-time analogue of the continuous-time Lie derivative defined by (9). For that purpose we will repeat the discrete-time counterparts of three steps in Section 2, leading to (9) and demonstrate that the result agrees with (26).

Consider the vector field (24) and examine its change caused by the forward shift from point x to point x^σ instead of the continuous flow. At the first step define, like in (5), a vector Ξ at the endpoint $x^\sigma = \Phi(x)$ by

$$\Xi(x^\sigma) = \sum_{i=1}^n \Xi_i(x^\sigma) \frac{\partial}{\partial x_i^\sigma}. \quad (37)$$

At the second step compute the change of the vector field Ξ . This step splits into two substeps. At the first substep one has to bring $\Xi(x)$ from the initial point x into the endpoint x^σ like in Section 2 (see e.g. [14], p. 62), multiplying it by the Jacobi matrix $T\Phi$ and rewriting the result $(T\Phi \cdot \Xi)(x)$, as a linear combination of basis vectors

$$\sum_{i=1}^n \left(\frac{\partial \Phi_i}{\partial x} \Xi \right) (x) \frac{\partial}{\partial x_i^\sigma}. \quad (38)$$

At the second substep we replace x by $\Phi^{-1}(x^\sigma)$, yielding

$$\sum_{i=1}^n \left(\frac{\partial \Phi_i}{\partial x} \Xi \right) (\Phi^{-1}(x^\sigma)) \frac{\partial}{\partial x_i^\sigma}.$$

Finally, the change of the vector field Ξ may be found as the difference

$$\Xi(x^\sigma) - (T\Phi \cdot \Xi) (\Phi^{-1}(x^\sigma)). \quad (39)$$

In the continuous-time case at the third step we divided the result by $\Delta t = \tau$, having no discrete-time analogue when $\Delta t = 1$, and “brought the vector back” to the initial point x by approaching the limit $\Delta t \rightarrow 0$. In the discrete-time case, the corresponding step is to evaluate (39) at the point x , yielding definition (26) of the backward Lie derivative.

To conclude, the backward Lie derivative of the vector field, defined by (26), is a discrete-time analogue of the continuous-time Lie derivative defined by (9). In a similar manner it may be shown that the forward Lie derivative of a vector field defined by (25) is a direct analogue of the alternative continuous-time Lie derivative given by (10).

4. THE GEOMETRIC PROPERTY OF THE LIE DERIVATIVE

The goal of this section is to show how the concept of the backward Lie derivative as defined by (26) is related to the geometric property of the vector field studied by Rieger, Schlacher, and Holl in [8], providing alternative interpretation for $L_{\Phi}^{\vee}\Xi$.

Consider, like in [8], the one-parameter group of transformations $\Psi_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$, generated by the vector field $\Xi = \sum_{i=1}^n \Xi_i(x) \partial/\partial x_i$. This group $\Psi_s = \exp(s\Xi)$, $s \in \mathbb{R}$, transforms the point $x \in \mathbb{R}^n$ into the point $\tilde{x} = \Psi_s(x)$; see Fig. 1, left vertical curve³.

The upper dotted line corresponds to the forward shift of the point x into the point $x^\sigma = \Phi(x)$ by system dynamics. The lowest dotted line shows the forward shift $\tilde{x}^\sigma = \Phi(\tilde{x})$ of the point \tilde{x} again by system dynamics. On the other hand, the flow Ψ_s generated by Ξ transfers the point x^σ into the point $\Psi_s(x^\sigma)$ (the right vertical curve). That is, the forward shift $x \rightarrow \Phi(x)$ has been transferred by the flow Ψ_s into a shift $\tilde{x} \rightarrow \tilde{\Phi}_s(\tilde{x})$, modified by the flow $\Psi_s = \exp(s\Xi)$ and called shortly the *modified forward shift* (the middle dotted line), which can be calculated by the composition

$$\tilde{\Phi}_s(\tilde{x}) := (\Psi_s \circ \Phi \circ \Psi_{-s})(\tilde{x}). \tag{40}$$

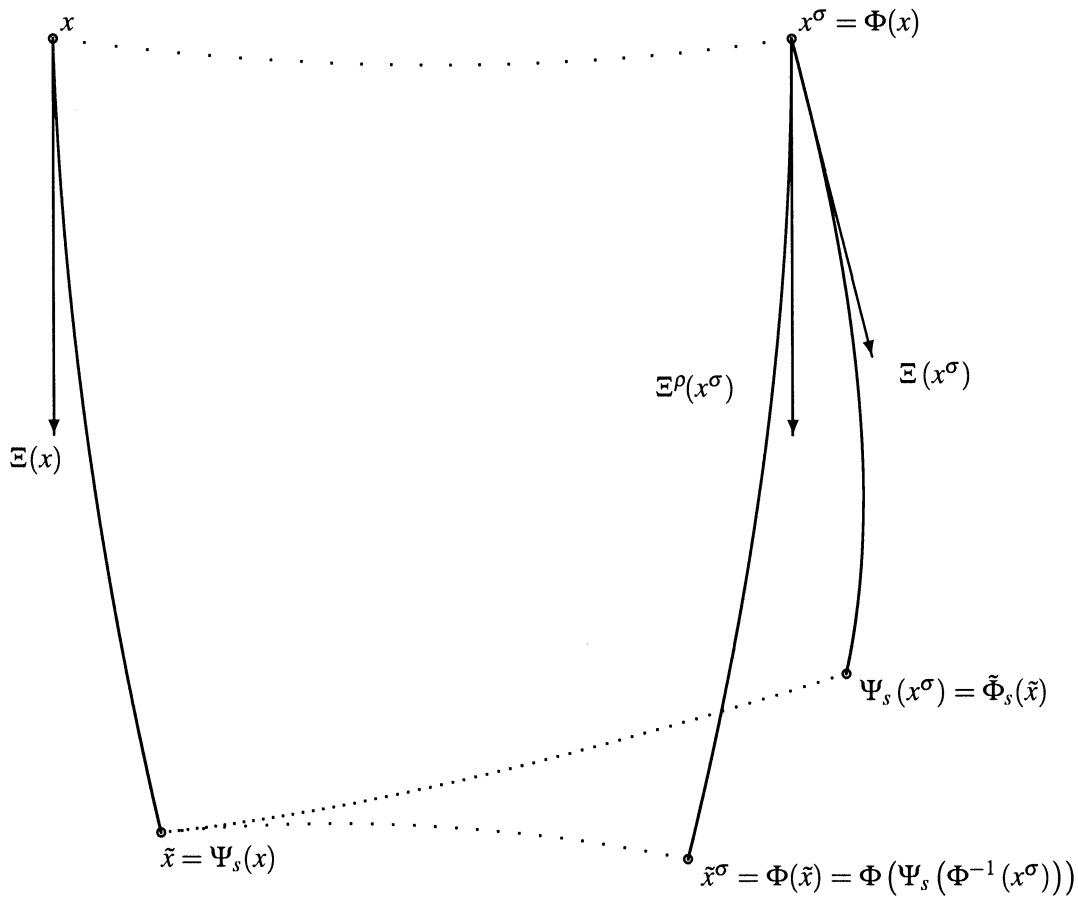


Fig. 1. The backward Lie derivative as the generator of the modified forward shift.

³ Figure 1 mimics the corresponding figure in [15] for the continuous-time case.

The diagram in Fig. 1 does not commute, in general, or said alternatively, the modified forward shift $\tilde{\Phi}_s(\tilde{x})$ of the point \tilde{x} does not coincide with $\Phi(\tilde{x})$, except when the difference

$$\delta_{s,x} := \Psi_s(\Phi(x)) - \Phi(\Psi_s(x)) \equiv 0, \tag{41}$$

i.e. if the flow Ψ_s commutes with the forward shift generated by Φ for an arbitrary $s \in \mathbb{R}$; see [8]. Note that according to the chain rule [8],

$$\left. \frac{\partial(\delta_{s,x})}{\partial s} \right|_{s=0} = \left(\left. \frac{\partial \Psi_s}{\partial s} \right|_{\Phi(x)} - \left. \frac{\partial \Phi}{\partial \Psi_s} \right|_{\Psi_s(x)} \left. \frac{\partial \Psi_s}{\partial s} \right|_x \right) \Big|_{s=0}. \tag{42}$$

Because the vector field Ξ is the infinitesimal generator of the flow Ψ_s , the first term on the right-hand side gives the vector Ξ evaluated at the point $x^\sigma = \Phi(x)$. The second term is the product of two factors. The first one is the Jacobi matrix $(\partial\Phi/\partial\Psi_s)$ evaluated at $\Psi_s(x)$, which for $s = 0$ when $\Psi_0(x) = x$, gives $(\partial\Phi/\partial x) = T\Phi$. The second factor is the vector Ξ defined at x . Consequently, we get

$$\left. \frac{\partial(\delta_{s,x})}{\partial s} \right|_{s=0} = \Xi(\Phi(x)) - \frac{\partial\Phi(x)}{\partial x} \Xi(x). \tag{43}$$

Continuing the study in [8], notice that the vector field (43) is defined at the point x^σ , therefore one has to express its components also in terms of x^σ , using the fact that $\Phi^{-1}(x^\sigma) = x$:

$$\left. \frac{\partial}{\partial s} [\delta_s(\Phi^{-1}(x^\sigma))] \right|_{s=0} = \Xi(x^\sigma) - \left(\frac{\partial\Phi}{\partial x} \Xi \right) (\Phi^{-1}(x^\sigma)). \tag{44}$$

Evaluating (44) at x results in the backward Lie derivative (26). It follows from the above discussion that if the flow Ψ_s , generated by Ξ , commutes with the forward shift Φ (see (41)), then $L_\Phi^\nabla \Xi \equiv 0$. This is in agreement with the corresponding property of the Lie derivative in the continuous-time case.

Next we examine the general case $L_\Phi^\nabla \Xi \neq 0$ and show that $L_\Phi^\nabla \Xi$ generates the transformation of \tilde{x}^σ into $\tilde{\Phi}_s(x)$, that is it transfers the forward shift of the point \tilde{x} into its modified forward shift. To show this, examine the terms on the right-hand side of (44) separately. As already known, $\Xi(x^\sigma)$ generates the motion of x^σ into the modified forward shift $\tilde{\Phi}_s(\tilde{x})$; see Fig. 1. We will prove that the second term in (44) which is, according to (30), just Ξ^ρ but defined at x^ρ , generates the motion of x^σ into the forward shift of $\tilde{x} = \Phi(\tilde{x})$. Note that $\tilde{x}^\sigma = \Phi(\Psi_s(\Phi^{-1}(x^\sigma)))$; see Fig. 1. Taking the partial derivative of this composition with respect to s at $s = 0$, we get, using the chain rule,

$$\left. \frac{\partial}{\partial s} [\Phi(\Psi_s(\Phi^{-1}(x^\sigma)))] \right|_{s=0} = \left. \frac{\partial\Phi}{\partial\Psi_s} \right|_{\Psi_s(\Phi^{-1}(x^\sigma))} \left. \frac{\partial\Psi_s}{\partial s} \right|_{\Phi^{-1}(x^\sigma), s=0} = \left. \frac{\partial\Phi}{\partial x} \right|_{\Phi^{-1}(x^\sigma)} \Xi(\Phi^{-1}(x^\sigma)) = \Xi^\rho(x^\sigma), \tag{45}$$

meaning that $\Xi^\rho(x^\sigma)$ is really the infinitesimal generator of the composition $\Phi(\Psi_s(\Phi^{-1}(x^\sigma)))$.

To conclude, the vector $\Xi(x^\sigma)$ generates the transfer of x^σ into the modified forward shift $\tilde{\Phi}_s(\tilde{x})$ of \tilde{x} . The vector $\Xi^\rho(x^\sigma)$ generates the transfer of x^σ into the forward shift $\Phi(\tilde{x})$ of \tilde{x} . Consequently, the backward Lie derivative $(L_\Phi^\nabla \Xi)(x^\sigma)$ as the difference of the vector field and its backward shift generates the transfer of \tilde{x}^σ into the modified forward shift $\tilde{\Phi}_s(\tilde{x})$ of \tilde{x} . In the special case where $L_\Phi^\nabla \Xi \equiv 0$, the forward shift of \tilde{x} equals its modified forward shift $\Phi(\tilde{x}) = \tilde{\Phi}_s(\tilde{x})$ for all $x \in \mathbb{R}^n$ and for all s .

5. EXAMPLES

The first example demonstrates how to compute the backward Lie derivatives of the vector fields.

Example 1. Consider the system

$$x_1^\sigma = x_2, \quad x_2^\sigma = x_1 x_2$$

whose Jacobi matrix is

$$T\Phi(x) = \begin{pmatrix} 0 & 1 \\ x_2 & x_1 \end{pmatrix}$$

and the inverse map Φ^{-1} , applied to the coordinates x , gives their backward shifts

$$x_1^\rho = \frac{x_2}{x_1}, \quad x_2^\rho = x_1. \quad (46)$$

Calculate, according to (26), the backward Lie derivatives of the basis vector fields. We start with $\partial/\partial x_1$ and calculate its Lie derivative by (26), (30), and (14).

First compute, according to (30),

$$\left(\frac{\partial}{\partial x_1}\right)^\rho = \left(T\Phi \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) (\Phi^{-1}(x)),$$

rewritten as a linear combination of the basis vectors

$$x_2^\rho \frac{\partial}{\partial x_2} = x_1 \frac{\partial}{\partial x_2},$$

and then by (26)

$$L_\Phi^\nabla \left(\frac{\partial}{\partial x_1}\right) (x) = \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$

In a similar manner one calculates the Lie derivative of $\partial/\partial x_2$:

$$L_\Phi^\nabla \left(\frac{\partial}{\partial x_2}\right) (x) = -\frac{\partial}{\partial x_1} + \left(1 - \frac{x_2}{x_1}\right) \frac{\partial}{\partial x_2}.$$

Compute also the backward Lie derivatives of a vector field $\Xi = x_1 \partial/\partial x_1 + x_2 \partial/\partial x_2$. Its backward shift, according to (30), is

$$\Xi^\rho = \left(T\Phi \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) (\Phi^{-1}(x)) = \begin{bmatrix} x_2 \\ 2x_1 x_2 \end{bmatrix} (\Phi^{-1}(x)),$$

rewritten as a linear combination of the basis vectors as

$$x_2^\rho \frac{\partial}{\partial x_1} + 2x_1^\rho x_2^\rho \frac{\partial}{\partial x_2} = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2},$$

and then by (26)

$$L_\Phi^\nabla \left(\frac{\partial}{\partial x_1}\right) (x) = -x_2 \frac{\partial}{\partial x_2}.$$

In the second example we show how the backward Lie derivative $L_{\Phi}^{\nabla}\Xi$ transfers the forward shift $\Phi(\tilde{x})$ of a point \tilde{x} into its modified forward shift $\tilde{\Phi}_s(\tilde{x})$, provided the vector field Ξ generates the flow Ψ_s and $\tilde{x} = \Psi_s(x)$, as described in Section 4.

Example 2. Consider the forward shift σ defined on \mathbb{R}^2 by the map Φ

$$\begin{aligned} x_1^{\sigma} &= x_1 \cos \phi - x_2 \sin \phi, \\ x_2^{\sigma} &= x_1 \sin \phi + x_2 \cos \phi, \end{aligned} \tag{47}$$

describing the rotation around the origin by an angle ϕ . The inverse Φ^{-1} of the map Φ is given by

$$\begin{aligned} x_1 &= x_1^{\sigma} \cos \phi + x_2^{\sigma} \sin \phi, \\ x_2 &= -x_1^{\sigma} \sin \phi + x_2^{\sigma} \cos \phi. \end{aligned}$$

Compute

$$T\Phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \tag{48}$$

The vector field $\Xi = \partial/\partial x_1$ generates the flow

$$\Psi_s : \quad \tilde{x}_1 = x_1 + s, \quad \tilde{x}_2 = x_2. \tag{49}$$

Calculate $\Xi^{\rho}(x^{\rho})$ as the column vector

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}.$$

Consequently

$$\Xi^{\rho}(x^{\sigma}) = \cos \phi \frac{\partial}{\partial x_1^{\rho}} + \sin \phi \frac{\partial}{\partial x_2^{\rho}}, \tag{50}$$

and by (26)

$$(L_{\Phi}^{\nabla}\Xi)(x^{\sigma}) = (1 - \cos \phi) \frac{\partial}{\partial x_1^{\sigma}} - \sin \phi \frac{\partial}{\partial x_2^{\sigma}}. \tag{51}$$

Figure 2 illustrates how the vector field $L_{\Phi}^{\nabla}\Xi$, defined by (51), transfers the point $\tilde{x}^{\sigma} = \Phi(\tilde{x})$ into the point $\tilde{\Phi}_s(\tilde{x})$ (the upper dotted line). That is, it shows the transfer from the *forward shift* of \tilde{x} into its *modified forward shift*, when the Ξ -generated flow Ψ_s acts in \mathbb{R}^n . The vector $s\Xi(x)$ transfers the point x with the coordinates $(x_1, 0)$ into the point \tilde{x} with the coordinates $(x_1 + s, 0)$. The forward shifts of the points x and \tilde{x} are, respectively, the point x^{σ} with coordinates $(x_1 \cos \phi, x_1 \sin \phi)$ (along the arc of the small circle), and the point $\tilde{x}^{\sigma} = \Phi(\tilde{x})$ with the coordinates $((x_1 + s) \cos \phi, (x_1 + s) \sin \phi)$ (along the arc of the large circle). The vector $s\Xi^{\rho}(x^{\sigma})$ transfers x^{σ} into \tilde{x}^{σ} .

Moreover, the vector $s\Xi(x^{\sigma})$ transfers x^{σ} as the forward shift of x into $\tilde{\Phi}_s(\tilde{x})$, which is the modified forward shift of \tilde{x} (marked by the lower dotted line) and has the coordinates $(s + x_1 \cos \phi, x_1 \sin \phi)$. Consequently, the difference of the vectors $s\Xi(x^{\sigma})$ and $s\Xi^{\rho}(x^{\sigma})$, which is $(L_{\Phi}^{\nabla}\Xi)(x^{\sigma})$, transfers the point $\tilde{x}^{\sigma} = \Phi(\tilde{x})$ into $\tilde{\Phi}_s(\tilde{x})$. The difference of the coordinates of these points is

$$\Phi(\tilde{x}) - \tilde{\Phi}_s(\tilde{x}) = [s(1 - \cos \phi), -s \sin \phi]^T, \tag{52}$$

which coincides with the components of $sL_{\Phi}^{\nabla}\Xi$ in (51). This means, the vector $(L_{\Phi}^{\nabla}\Xi)(x^{\sigma})$ really transfers the forward shift of x into its modified forward shift.

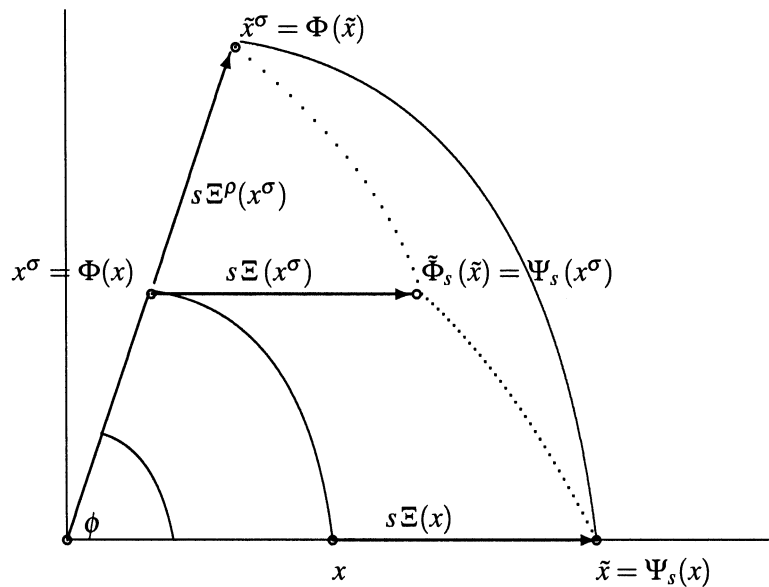


Fig. 2. The modification of a rotation by forcing it with the backward Lie derivative.

6. CONCLUSION

In this paper the concept of the Lie derivative of a vector field, applied frequently in the study of continuous-time dynamical systems, has been extended to the discrete-time case. Actually, in the discrete-time domain the single concept branches into two concepts, backward and forward Lie derivatives, depending on whether one applies in its definition the forward or backward difference (see formulae (9) and (10)). The introduced algebraic definition of the backward Lie derivative agrees with the discrete-time analogue of its continuous-time interpretation and also satisfies the geometric property of the Lie derivative known from the continuous-time theory. The interpretation of the forward Lie derivative also agrees with the continuous-time interpretation.

Our future goals are to extend the results of this paper (i) for discrete-time nonlinear control systems and (ii) for systems defined on homogeneous time scale. The first steps towards (i) were made in [16]. However, note that in [16] we took off from the continuous-time interpretation of the Lie derivative as recalled in Section 2 and repeated the discrete-time counterparts of the respective steps to get the definition in discrete time. In doing so we overlooked the algebraic meaning of the definition. For example, we did not recognize that the so-called Θ -operator in [16] is actually a backward shift operator. Note that the algebraic definition of Lie derivative for control systems requires extension of the difference field, associated with the state equations, up to its inversive closure. This aspect was also not worked out fully in [16].

Once the Lie derivative is defined either for discrete-time control systems or for control systems on time scales, it possibly allows extension of the numerous results/methods/algorithms that are based on the concept of the Lie derivative into the discrete-time or time scale domain. The problems addressed in terms of Lie derivatives include for example accessibility [11], feedback linearization [13], realization in state-space form [17–20], and lowering the maximal order of the input time derivatives in the generalized state equations [18].

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Lie tuletise üldistus diskreetajaga süsteemidele

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Pideva ajaga süsteemis on skalaarfunktsiooni, vektorvälja või 1-vormi Lie tuletis süsteemi dünaamika suhtes defineeritud tema muutumise kiirusena ajas. Artikkel annab esmalt vektorvälja Lie tuletise mõiste algebralise definitsiooni diskreetaja süsteemidele, kasutades otsest analoogiat funktsiooni muuduga. Lie tuletise arvutamiseks vajalikud vektorvälja edasi- ja tagasinihked defineeritakse juba teadaolevate 1-vormi edasi- ning tagasinihete ja 1-vormi ning vektorvälja skalaarkorrutise abil. Järgnevalt näidatakse, et sellisel defineeritud vektorvälja diskreetaja Lie tuletis on tema muut, mis vastab ajakoordinaadi kasvule ühe võrra, olles otsene analoog vektorvälja muutumise kiirusele ajas. Samuti uuritakse lähemalt vektorvälja diskreetaja Lie tuletise geomeetrilisi omadusi.