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Algebraic formalism of differential p-forms and vector fields for nonlinear control systems on homogeneous time scales

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Abstract. The paper develops further the algebraic formalism for nonlinear control systems defined on homogeneous time scales. The delta derivative operator is extended to differential p-forms and vector fields.

Key words: time scale, nonlinear system, delta derivative, differential p-forms, vector fields.

1. INTRODUCTION

Algebraic formalism for nonlinear control systems based on differential one-forms has been developed separately for continuous-time systems [1] and discrete-time systems [2–4]. In [5] a single common formalism based on time scale calculus has been introduced. It covers both the continuous- and discrete-time cases in such a manner that those are the special cases of the formalism. However, it has to be stressed that in [5] the discrete-time system is described in terms of the difference operator unlike in the majority of papers where the system is described via the shift operator (see for example [2–4,6,7]).

This paper may be understood as the continuation of paper [5], which developed the algebraic formalism of differential one-forms associated with the nonlinear control system defined on a homogeneous time scale. An inversive σ_f-differential field $\mathscr{K}$ of meromorphic functions in system variables equipped with two operators, delta derivative Δ_f, and forward jump σ_f, was constructed under the nonrestrictive assumption which guarantees the submersivity of the system. In the continuous-time case the delta derivative is just an ordinary time derivative and the forward jump is an identity operator. In the discrete-time case the delta derivative is the forward difference and the forward jump is the forward shift operator. Moreover, a vector space $\mathscr{E}$ (over $\mathscr{K}$) of differential one-forms was introduced, the operators Δ_f and σ_f were extended to $\mathscr{E}$ and some of their properties were studied. The developed formalism has later been used in [8–11] to study different modelling and analysis problems. In [12] the results of [5] were partly extended for the case of regular but nonhomogeneous time scales. In the case of nonhomogeneous time scales the delta derivative and shift operator do not commute. However, the main difficulty is related to the fact that the additional time

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variable t appears in functions from the differential rings associated with the control system. Therefore, in the construction of the inversive closure of the differential ring the new variables depend on t and have to be chosen to be smooth at each dense point of the time scale (see [12]). Moreover, the differential ring associated with the considered system can have zero devisors, so it is impossible to construct its quotient field (see [12]). One can easily observe that taking the nonhomogeneous time scale the graininess function which depends on point t from time scale $\mathbb{T}$ may not be continuous and consequently not delta differentiable, so delta differentiability of the graininess function is the problem that one can encounter for nonhomogeneous time scales. Then the computation of the higher-order delta derivatives of functions is not always possible.

The goal of this paper is to unify the calculus of p-forms, extend the operators of the delta derivative and forward jump to p-forms and prove some of their properties. Moreover, we also introduce the dual space of vector fields and extend the operators of the delta derivative and forward jump to vector fields.

2. TIME SCALE CALCULUS AND DIFFERENTIAL FIELD

For a general introduction to the calculus on time scales, see [13]. Here we recall only those notions and facts that will be used later.

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We assume that the topology of $\mathbb{T}$ is induced by $\mathbb{R}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$, $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, the *backward jump operator* $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$, $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. The graininess functions $\mu : \mathbb{T} \to [0,\infty)$ and $\nu : \mathbb{T} \to [0,\infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$, respectively. A time scale is called *homogeneous* if μ and ν are constant functions.

Let $\mathbb{T}^\kappa$ denote a truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point. The reason for omitting a maximal left-scattered point is to guarantee uniqueness of f^Δ, defined below.

Definition 2.1. *The delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}^\kappa$ is the real number $f^\Delta(t)$ (provided it exists) such that for each $\varepsilon > 0$ there exists a neighbourhood $U(\varepsilon)$ of t, $U(\varepsilon) \subset \mathbb{T}$ such that for all $\tau \in U(\varepsilon)$, $|(f(\sigma(t)) - f(\tau)) - f^\Delta(t)(\sigma(t) - \tau)| \leqslant \varepsilon|\sigma(t) - \tau|$. Moreover, we say that f is* delta differentiable *on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.*

For a function $f : \mathbb{T} \to \mathbb{R}$ we can define the second delta derivative $f^{[2]} := \left(f^\Delta\right)^\Delta$ provided that f^Δ is delta differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$ with the derivative $f^{[2]} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly we define higher-order delta derivatives $f^{[n]} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$, $f^{[n]} := \left(f^{[n-1]}\right)^\Delta$, where $\mathbb{T}^{\kappa^n} = \left(\mathbb{T}^{\kappa^{n-1}}\right)^\kappa$, $n \geqslant 1$. Note that for homogeneous time scale $\mathbb{T}^\kappa = \mathbb{T}$, i.e. there is no left-scattered maximal point in $\mathbb{T}$, so $f^{[n]}$, $n \geqslant 1$ are uniquely defined for all $t \in \mathbb{T}$. From now on we assume that $\mathbb{T}$ is homogeneous.

For $f : \mathbb{T} \to \mathbb{R}$ define $f^\sigma := f \circ \sigma$. Denote $f^{\Delta\sigma} := \left(f^\Delta\right)^\sigma$ and $f^{\sigma\Delta} := (f^\sigma)^\Delta$.

If f and f^Δ are delta differentiable functions, then for homogeneous time scale one has $f^{\sigma\Delta} = f^{\Delta\sigma}$.

Consider now the control system, defined on a homogeneous time scale $\mathbb{T}$,

$$x^\Delta(t) = f(x(t), u(t)), \tag{1}$$

where $(x(t), u(t)) \in U$, U is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, $m \leqslant n$, and $f : U \to \mathbb{R}^n$ is analytic. Let us define $\widetilde{f}(x,u) := x + \mu f(x,u)$ and assume[1] that there exists a map $\varphi : U \to \mathbb{R}^m$ such that $\Phi = (\widetilde{f}, \varphi)^T$ is an analytic diffeomorphism from the set U onto U. This means that from $(\bar{x}, z) = \left(\widetilde{f}(x,u), \varphi(x,u)\right) = \Phi(x,u)$ we can uniquely compute (x,u) as an analytic function of $(\bar{x}, z)$. For $\mu = 0$ this condition is always satisfied with $\varphi(x,u) = u$.

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[1] This assumption guarantees that the system $x^\sigma = \tilde{f}(x,u)$ is submersive, that is generically rank $\frac{\partial \tilde{f}(x,u)}{\partial(x,u)} = n$.

The basic types of operators covered by the delta derivative are the time derivative, and the difference operator $[f(t+1)-f(t)]/\mu$. So, in the special case $\mathbb{T}=\mathbb{R}$, equation (1) becomes an ordinary differential equation and when $\mathbb{T}=hZ$, for $h>0$, it becomes the difference equation.

For notational convenience, $(x_1,\ldots,x_n)$ will simply be written as x, and $\left(u_1^{[k]},\ldots,u_m^{[k]}\right)$ as $u^{[k]}$, for $k\geqslant 0$. For $i\leqslant k$, let $u^{[i\ldots k]}:=\left(u^{[i]},\ldots,u^{[k]}\right)$. We assume that the input applied to system (1) is infinitely many times delta differentiable, i.e. $u^{[0\ldots k]}$ exists for all $k\geqslant 0$. Consider the infinite set of real (independent) indeterminates

$$\mathscr{C}=\left\{x_i, i=1,\ldots,n,\ u_j^{[k]}, j=1,\ldots,m,\ k\geqslant 0\right\}$$

and let $\mathscr{K}$ be the (commutative) field of meromorphic functions in a finite number of the variables from the set $\mathscr{C}$. Let $\sigma_f:\mathscr{K}\to\mathscr{K}$ be an operator defined by

$$\sigma_f(F)\left(x,u^{[0\ldots k+1]}\right):=\varphi\left(x^{\sigma},\left(u^{[0\ldots k]}\right)^{\sigma}\right),$$

where $F\in\mathscr{K}$ depends on x and $u^{[0\ldots k]}$, $\left(u^{[0\ldots k]}\right)^{\sigma}=u^{[0\ldots k]}+\mu u^{[1\ldots k+1]}$, for $k\geqslant 0$ and by (1) $x^{\sigma}=x+\mu x^{\Delta}=x+\mu f(x,u)$. We assume that $(x,u)\in U$ and the other variables are restricted in such a way that σ_f is well defined. Under the assumption about the existence of φ such that $\Phi=\left(\widetilde{f},\varphi\right)$ is an analytic diffeomorphism, σ_f is an injective endomorphism.

The field $\mathscr{K}$ can be equipped with a delta derivative operator $\Delta_f:\mathscr{K}\to\mathscr{K}$ defined by

$$\Delta_f(F)\left(x,u^{[0\ldots k+1]}\right)=\begin{cases}\dfrac{1}{\mu}\left[F\left(x+\mu f(x,u),u^{[0\ldots k]}+\mu u^{[1\ldots k+1]}\right)-F\left(x,u^{[0\ldots k]}\right)\right], & \text{if } \mu\neq 0\\[2ex] \dfrac{\partial F}{\partial x}\left(x,u^{[0\ldots k]}\right)f(x,u)+\displaystyle\sum_{k\geqslant 0}\dfrac{\partial F}{\partial u^{[0\ldots k]}}\left(x,u^{[0\ldots k]}\right)u^{[1\ldots k+1]}, & \text{if } \mu=0,\end{cases}\tag{2}$$

where $F\in\mathscr{K}$ depends on x and $u^{[0\ldots k]}$.

The more compact notations F^{σ_f} and F^{Δ_f} will be sometimes used instead of $\sigma_f(F)$ and $\Delta_f(F)$.

The delta derivative Δ_f satisfies, for all $F,G\in\mathscr{K}$, the conditions

(i) $\Delta_f(F+G)=\Delta_f(F)+\Delta_f(G)$,

(ii) $\Delta_f(FG)=\Delta_f(F)G+\sigma_f(F)\Delta_f(G)$ (generalized Leibniz rule).

An operator satisfying the generalized Leibniz rule is called a "*σ_f-derivation*" and a commutative field endowed with a σ_f-derivation is called a *σ_f-differential field* [14]. Therefore, under the assumption about the existence of φ such that $\Phi=\left(\widetilde{f},\varphi\right)$ is an analytic diffeomorphism, $\mathscr{K}$ endowed with the delta derivative Δ_f is a σ_f-differential field. For $\mu=0$, $\sigma_f=\sigma_f^{-1}=\mathrm{id}$ and $\mathscr{K}$ is inversive. Although $\mathscr{K}$ is not inversive in general, it is always possible to embed $\mathscr{K}$ into an inversive σ_f-differential overfield $\mathscr{K}^*$, called the *inversive closure* of $\mathscr{K}$ [14]. Since σ_f is an injective endomorphism, it can be extended to $\mathscr{K}^*$ so that $\sigma_f:\mathscr{K}^*\to\mathscr{K}^*$ is an automorphism. It was shown in [5] that for $\mu\neq 0$ the inversive closure of $\mathscr{K}$ may be constructed as the field of meromorphic functions in a finite number of the independent variables $\mathscr{C}^*=\mathscr{C}\cup\{z_s^{\langle-\ell\rangle},\ s=1,\ldots,m,\ \ell\geqslant 1\}$, where the new variables are related by σ_f as follows: $z_i^{\langle-k\rangle}=\sigma_f\left(z_i^{\langle-k-1\rangle}\right)$ and $z_i=\varphi_i(x,u)=\sigma_f\left(z_i^{\langle-1\rangle}\right)$. Let $z:=(z_1,\ldots,z_m)$. Then $\sigma_f^{-1}(x,u)=\psi\left(x,z^{\langle-1\rangle}\right)$, where ψ is a certain vector-valued function, determined by f in (1) and the extension $z=\varphi(x,u)$. Although the choice of variables z is not unique, all possible choices yield isomorphic field extensions. We extend the operator Δ_f to new variables by

$$\Delta_f(z^{\langle-\ell\rangle}):=\frac{z^{\langle-\ell+1\rangle}-z^{\langle-\ell\rangle}}{\mu},\quad l\geq 1.$$

The extension of operator Δ_f to $\mathscr{K}^*$ can be made in analogy to (2). Such operator Δ_f is now a σ_f-derivation of $\mathscr{K}^*$. A practical procedure for the construction of $\mathscr{K}^*$ (for $\mu \neq 0$) is given in [5].

From now on

$$\mathscr{C}^* = \begin{cases} \mathscr{C}, & \text{if } \mu = 0 \\ \mathscr{C} \cup \{z^{\langle -\ell \rangle} \mid \ell \geqslant 1\}, & \text{if } \mu \neq 0. \end{cases}$$

Consider the infinite set of symbols $\mathrm{d}\mathscr{C}^* = \{\mathrm{d}\zeta_i,\ \zeta_i \in \mathscr{C}^*\}$ and define $\mathscr{E} := \mathrm{span}_{\mathscr{K}^*}\mathrm{d}\mathscr{C}^*$. Any element of $\mathscr{E}$ is a vector of the form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k\geqslant 0}\sum_{j=1}^{m} B_{jk}\mathrm{d}u_j^{[k]} + \sum_{\ell\geqslant 1}\sum_{s=1}^{m} C_{s\ell}\mathrm{d}z_s^{\langle -\ell \rangle},$$

where only a finite number of coefficients B_{jk} and $C_{s\ell}$ are nonzero elements of $\mathscr{K}^*$.

The elements of $\mathscr{E}$ are called differential *one-forms*. Let $\mathrm{d} : \mathscr{K}^* \to \mathscr{E}$ be defined in the standard manner:

$$\mathrm{d}F := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\mathrm{d}x_i + \sum_{k\geqslant 0}\sum_{j=1}^{m} \frac{\partial F}{\partial u_j^{[k]}}\mathrm{d}u_j^{[k]} + \sum_{\ell\geqslant 1}\sum_{s=1}^{m} \frac{\partial F}{\partial z_s^{\langle -\ell \rangle}}\mathrm{d}z_s^{\langle -\ell \rangle}. \tag{3}$$

One says that $\omega \in \mathscr{E}$ is an *exact one-form* if $\omega = \mathrm{d}F$ for some $F \in \mathscr{K}^*$. We will refer to $\mathrm{d}F$ as to the *total differential* (or simply the *differential*) of F.

If $\omega = \sum_i A_i \mathrm{d}\zeta_i$ is a one-form, where $A_i \in \mathscr{K}^*$ and $\zeta_i \in \mathscr{C}^*$, one can define the operators $\Delta_f : \mathscr{E} \to \mathscr{E}$ and $\sigma_f : \mathscr{E} \to \mathscr{E}$ by

$$\Delta_f(\omega) := \sum_i \left\{ \Delta_f(A_i)\mathrm{d}\zeta_i + \sigma_f(A_i)\mathrm{d}\left[\Delta_f(\zeta_i)\right] \right\},$$

and

$$\sigma_f(\omega) := \sum_i \sigma_f(A_i)\mathrm{d}\left[\sigma_f(\zeta_i)\right].$$

The operator $\sigma_f : \mathscr{E} \to \mathscr{E}$ is invertible and the inverse operator $\rho_f := \sigma_f^{-1} : \mathscr{E} \to \mathscr{E}$ is defined by

$$\sigma_f^{-1}\left(\sum_i A_i \mathrm{d}\zeta_i \right) = \sum_i \sigma_f^{-1}(A_i)\,\mathrm{d}\left[\sigma_f^{-1}(\zeta_i)\right],$$

for $A_i \in \mathscr{K}^*$ and $\zeta_i \in \mathscr{C}^*$.

Since $\sigma_f(A_i) = A_i + \mu\Delta_f(A_i)$,

$$\Delta_f(\omega) = \sum_i \left\{ \Delta_f(A_i)\mathrm{d}\zeta_i + (A_i + \mu\Delta_f(A_i))\,\mathrm{d}\left[\Delta_f(\zeta_i)\right] \right\}.$$

It was shown in [5] that for the homogeneous time scale $\mathbb{T}$ we have

$$\begin{aligned} \Delta_f\left[\mathrm{d}F\right] &= \mathrm{d}\left[F^{\Delta_f}\right], \\ \sigma_f\left[\mathrm{d}F\right] &= \mathrm{d}\left[F^{\sigma_f}\right], \end{aligned} \tag{4}$$

where $F \in \mathscr{K}^*$.

For one-forms similarly as for functions the more compact notations ω^{Δ_f} and ω^{σ_f} will be used instead of $\Delta_f(\omega)$ and $\sigma_f(\omega)$.

3. THE DUAL SPACE OF VECTOR FIELDS

Let $\mathscr{E}'$ be the dual vector space of $\mathscr{E}$, i.e. the space of linear mappings from $\mathscr{E}$ to $\mathscr{K}^*$. The elements of $\mathscr{E}'$ are of the form

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{k\geqslant 0}\sum_{j=1}^{m} b_{jk}\frac{\partial}{\partial u_j^{[k]}} + \sum_{\ell\geqslant 0}\sum_{s=1}^{m} c_{s\ell}\frac{\partial}{\partial z_s^{\langle -\ell\rangle}}\,, \tag{5}$$

where $a_i, b_{jk}, c_{s\ell} \in \mathscr{K}^*$ and are called the vector fields. Taking $\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k=0}^{p}\sum_{j=1}^{m} B_{jk}\mathrm{d}u_j^{[k]} + \sum_{\ell=1}^{q}\sum_{s=1}^{m} C_{s\ell}\mathrm{d}z_s^{\langle -\ell\rangle} \in \mathscr{E}$ and the vector field $X \in \mathscr{E}'$ of the form (5), we get

$$X(\omega) =: \langle X, \omega\rangle = \sum_{i=1}^{n} a_i A_i + \sum_{k=0}^{p}\sum_{j=1}^{m} b_{jk}B_{jk} + \sum_{\ell=1}^{q}\sum_{s=1}^{m} c_{s\ell}C_{s\ell}\,. \tag{6}$$

Note that even if the linear combination (5) is infinite, it nevertheless defines an element of $\mathscr{E}'$ because, for all $\omega \in \mathscr{E}$, $\langle X, \omega\rangle$ may be written as a sum with only finitely many nonzero terms; see (6).

The delta-derivative X^{Δ_f} and forward-shift X^{σ_f} of $X \in \mathscr{E}'$ may be defined uniquely by the equations

$$\langle X^{\Delta_f}, \omega\rangle = \langle X, \sigma_f^{-1}(\omega)\rangle^{\Delta_f} - \left\langle X, \left[\sigma_f^{-1}(\omega)\right]^{\Delta_f}\right\rangle \tag{7}$$

and

$$\langle X^{\sigma_f}, \omega\rangle = \langle X, \sigma_f^{-1}(\omega)\rangle^{\sigma_f}, \tag{8}$$

respectively, where ω is an arbitrary one-form. Note that $\langle X, \sigma_f^{-1}(\omega)\rangle \in \mathscr{K}^*$, so $\langle X, \sigma_f^{-1}(\omega)\rangle^{\sigma_f}$ and $\langle X, \sigma_f^{-1}(\omega)\rangle^{\Delta_f}$ are well defined.

Evaluating (7) and (8) with the elements of canonical basis (i.e. with the elements from the set $\mathrm{d}\mathscr{C}^*$), we obtain two systems of equations that define X^{Δ_f} and X^{σ_f}, respectively.

Proposition 3.1. *Let $X \in \mathscr{E}'$. Then for arbitrary $\omega \in \mathscr{E}$*

$$\begin{aligned} X^{\sigma_f} &= X + \mu X^{\Delta_f}, \\ \langle X, \omega\rangle^{\Delta_f} &= \langle X^{\Delta_f}, \omega\rangle + \langle X^{\sigma_f}, \omega^{\Delta_f}\rangle\,. \end{aligned} \tag{9}$$

Proof. Let $X \in \mathscr{E}'$. Note that $\langle X, \omega\rangle \in \mathscr{K}^*$ for arbitrary $\omega \in \mathscr{E}$ and

$$\langle X, \omega\rangle^{\sigma_f} = \langle X, \omega\rangle + \mu\langle X, \omega\rangle^{\Delta_f}\,.$$

Then by (7) and (8) we get

$$\begin{aligned} \langle X^{\sigma_f}, \omega^{\sigma_f}\rangle = \langle X, \omega\rangle^{\sigma_f} &= \langle X, \omega\rangle + \mu\left[\langle X^{\Delta_f}, \omega^{\sigma_f}\rangle + \langle X, \omega^{\Delta_f}\rangle\right] \\ &= \langle X, \omega^{\sigma_f}\rangle + \mu\langle X^{\Delta_f}, \omega^{\sigma_f}\rangle = \langle X + \mu X^{\Delta_f}, \omega^{\sigma_f}\rangle\,. \end{aligned} \tag{10}$$

From the invertibility of operator $\sigma_f : \mathscr{E} \to \mathscr{E}$ we get $\sigma_f(\mathscr{E}) = \mathscr{E}$. Therefore by (10) the relation (9) holds.

Moreover, from (7) and (9) we get

$$\begin{aligned} \langle X, \omega\rangle^{\Delta_f} &= \langle X^{\Delta_f}, \omega^{\sigma_f}\rangle + \langle X, \omega^{\Delta_f}\rangle = \langle X^{\Delta_f}, \omega\rangle + \mu\langle X^{\Delta_f}, \omega^{\Delta_f}\rangle + \langle X, \omega^{\Delta_f}\rangle \\ &= \langle X^{\Delta_f}, \omega\rangle + \langle X + \mu X^{\Delta_f}, \omega^{\Delta_f}\rangle = \langle X^{\Delta_f}, \omega\rangle + \langle X^{\sigma_f}, \omega^{\Delta_f}\rangle\,, \end{aligned}$$

for arbitrary $\omega \in \mathscr{E}$. □

We show below on a simple example how to compute X^{Δ_f} and X^{σ_f}.

Example 3.2. Consider the system described by

$$\begin{aligned} x_1^\Delta &= x_1 x_2, \\ x_2^\Delta &= u, \end{aligned}$$

defined on $U = \{(x_1, x_2, u) \in \mathbb{R}^3 : 1 + \mu x_2 > 0, u > 0\}$.

For $\mu > 0$ the system can be rewritten as

$$\begin{aligned} x_1^\sigma &= x_1 + \mu x_1 x_2, \\ x_2^\sigma &= x_2 + \mu u. \end{aligned}$$

Then the inversive closure of $\mathscr{K}$ can be chosen as the field of meromorphic functions in a finite number of variables x_1, x_2, $u^{[k]}$, $z^{\langle -\ell \rangle}$, $k \geqslant 0$, $\ell \geqslant 1$, where $z^{\langle -1 \rangle} = \sigma_f^{-1}(z)$ and $z^{\langle -\ell \rangle} = \sigma_f^{-1}(z^{\langle -\ell+1 \rangle})$.

We construct below the field extension in three different ways (choosing z as u, x_2 or x_1, respectively) and compute X^{Δ_f} and X^{σ_f} on different canonical bases of $\mathscr{E}'$, corresponding to three choices of the variable z.

Let $X = \frac{\partial}{\partial x_2}$ be an element of $\mathscr{E}'$. Note that X^{Δ_f} and X^{σ_f} have the following forms:

$$X^{\Delta_f} = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \sum_{j \geqslant 0} b_j \frac{\partial}{\partial u^{[j]}} + \sum_{\ell \geqslant 1} c_\ell \frac{\partial}{\partial z^{\langle -\ell \rangle}} \tag{11}$$

and

$$X^{\sigma_f} = \tilde{a}_1 \frac{\partial}{\partial x_1} + \tilde{a}_2 \frac{\partial}{\partial x_2} + \sum_{j \geqslant 0} \tilde{b}_j \frac{\partial}{\partial u^{[j]}} + \sum_{\ell \geqslant 1} \tilde{c}_\ell \frac{\partial}{\partial z^{\langle -\ell \rangle}}\,. \tag{12}$$

Case 1 ($z = u$). Since X^{Δ_f}, X^{σ_f} have the form (11), (12), respectively, $\sigma_f(\mathrm{d}x_i) = \mathrm{d}x_i^\sigma$ and $\Delta_f(\mathrm{d}x_i) = \mathrm{d}x_i^\Delta$, $i = 1, 2$,

$$\langle X, \mathrm{d}x_1 \rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}x_1^\sigma \rangle + \langle X, \mathrm{d}x_1^\Delta \rangle = (1 + \mu x_2) a_1 + \mu x_1 a_2 + x_1, \tag{13}$$

$$\langle X, \mathrm{d}x_2 \rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}x_2^\sigma \rangle + \langle X, \mathrm{d}x_2^\Delta \rangle = a_2 + \mu b_0, \tag{14}$$

$$\langle X, \mathrm{d}u^{[k]} \rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}(u^{[k]} + \mu u^{[k+1]}) \rangle + \langle X, \mathrm{d}u^{[k+1]} \rangle = b_k + \mu b_{k+1}, \quad k \geqslant 0, \tag{15}$$

$$\langle X, \mathrm{d}u^{\langle -1 \rangle} \rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}u \rangle + \left\langle X, \mathrm{d}\left(\frac{u - u^{\langle -1 \rangle}}{\mu} \right) \right\rangle = b_0, \tag{16}$$

$$\langle X, \mathrm{d}u^{\langle -\ell-1 \rangle} \rangle^{\Delta_f} = \quad 0 \quad = \left\langle X^{\Delta_f}, \mathrm{d}\left(u^{\langle -\ell \rangle} \right) \right\rangle + \left\langle X, \mathrm{d}\left(\frac{1}{\mu} \left[u^{\langle -\ell \rangle} - u^{\langle -\ell-1 \rangle} \right] \right) \right\rangle = c_\ell, \quad \ell \geqslant 1 \tag{17}$$

and

$$\langle X, \mathrm{d}x_1 \rangle^{\sigma_f} = \quad 0 \quad = \langle X^{\sigma_f}, \mathrm{d}x_1^\sigma \rangle = (1 + \mu x_2) \tilde{a}_1 + \mu x_1 \tilde{a}_2, \tag{18}$$

$$\langle X, \mathrm{d}x_2 \rangle^{\sigma_f} = \quad 1 \quad = \langle X^{\sigma_f}, \mathrm{d}x_2^\sigma \rangle = \tilde{a}_2 + \mu \tilde{b}_0, \tag{19}$$

$$\langle X, \mathrm{d}u^{[k]} \rangle^{\sigma_f} = \quad 0 \quad = \langle X^{\sigma_f}, \mathrm{d}(u^{[k]} + \mu u^{[k+1]}) \rangle = \tilde{b}_k + \mu \tilde{b}_{k+1}, \quad k \geqslant 0, \tag{20}$$

$$\langle X, \mathrm{d}u^{\langle -1 \rangle} \rangle^{\sigma_f} = \quad 0 \quad = \langle X^{\sigma_f}, \mathrm{d}u \rangle = \tilde{b}_0, \tag{21}$$

$$\langle X, \mathrm{d}u^{\langle -\ell-1 \rangle} \rangle^{\sigma_f} = \quad 0 \quad = \left\langle X^{\sigma_f}, \mathrm{d}\left(u^{\langle -\ell \rangle} \right) \right\rangle = \tilde{c}_\ell, \quad \ell \geqslant 1\,. \tag{22}$$

Then $b_k = c_\ell = 0$, for $k \geqslant 0$, $\ell \geqslant 1$, $a_1 = -\frac{x_1}{1+\mu x_2}$, $a_2 = 0$ and $\tilde{b}_k = \tilde{c}_\ell = 0$, for $k \geqslant 0$, $\ell \geqslant 1$, $\tilde{a}_1 = \frac{-\mu x_1}{1+\mu x_2}$, $\tilde{a}_2 = 1$. Therefore

$$X^{\Delta_f} = -\frac{x_1}{1+\mu x_2}\frac{\partial}{\partial x_1} \quad \text{and} \quad X^{\sigma_f} = -\frac{\mu x_1}{1+\mu x_2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2};$$

in particular, for $\mu = 0$ we have

$$X^{\Delta_f} = -x_1\frac{\partial}{\partial x_1} \quad \text{and} \quad X^{\sigma_f} = \frac{\partial}{\partial x_2}. \tag{23}$$

Case 2 ($z = x_2$). Alternatively, the inversive closure can be chosen as a field of meromorphic functions in a finite number of variables x_1, x_2, $u^{[k]}$, $x_2^{\langle -\ell \rangle}$, $k \geqslant 0$, $\ell \geqslant 1$, where $x_2^{\langle -1 \rangle} = \sigma_f^{-1}(x_2)$ and $x_2^{\langle -\ell \rangle} = \sigma_f^{-1}(x_2^{\langle -\ell+1 \rangle})$.

Since X^{Δ_f} has the form (11), $\sigma_f(\mathrm{d}x_i) = \mathrm{d}x_i^{\sigma}$ and $\Delta_f(\mathrm{d}x_i) = \mathrm{d}x_i^{\Delta}$, $i = 1,2$, taking $\langle X, \mathrm{d}x_1 \rangle^{\Delta_f}$, $\langle X, \mathrm{d}x_2 \rangle^{\Delta_f}$, $\langle X, \mathrm{d}u^{[k]} \rangle^{\Delta_f}$, $k \geqslant 0$, we get equations (13), (14), (15), respectively. For the considered vector field X and the differential one-form corresponding to new variable $x_2^{\langle -1 \rangle}$ we have

$$\langle X, \mathrm{d}x_2^{\langle -1 \rangle} \rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}x_2 \rangle + \left\langle X, \mathrm{d}\left(\frac{x_2 - x_2^{\langle -1 \rangle}}{\mu}\right)\right\rangle = a_2 + \frac{1}{\mu},$$

which is different from (16) given in Case 1 for the vector field X and the differential of new variable $u^{\langle -1 \rangle}$, but taking $\langle X, \mathrm{d}x_2^{\langle -\ell-1 \rangle} \rangle^{\Delta_f}$, $\ell \geqslant 1$, we get equations (17), i.e. $c_\ell = 0$, $\ell \geqslant 1$. Similarly, since X^{σ_f} have the form (12) and $\sigma_f(\mathrm{d}x_i) = \mathrm{d}x_i^{\sigma}$, $i = 1,2$, for $\langle X, \mathrm{d}x_1 \rangle^{\sigma_f}$, $\langle X, \mathrm{d}x_2 \rangle^{\sigma_f}$, $\langle X, \mathrm{d}u^{[k]} \rangle^{\sigma_f}$, $k \geqslant 0$, we have equations (18), (19), (20), respectively. For $\langle X, \mathrm{d}x_2^{\langle -1 \rangle} \rangle^{\sigma_f}$ we have

$$\langle X, \mathrm{d}x_2^{\langle -1 \rangle} \rangle^{\sigma_f} = \quad 0 \quad = \langle X^{\sigma_f}, \mathrm{d}x_2 \rangle = \tilde{a}_2, \tag{24}$$

which is different from (21) in Case 1, but taking $\langle X, \mathrm{d}x_2^{\langle -\ell-1 \rangle} \rangle^{\sigma_f}$, $\ell \geqslant 1$, we get equations (22), i.e. $\tilde{c}_\ell = 0$, $\ell \geqslant 1$. Then, as in Case 1, we get $c_\ell = 0$ and $\tilde{c}_\ell = 0$, for $\ell \geqslant 1$, but

$$b_k = \begin{cases} \frac{(-1)^k}{\mu^{k+2}}, & \text{if } \mu \neq 0 \\ 0, & \text{if } \mu = 0 \end{cases}, \quad \text{for} \quad k \geqslant 0, \quad a_1 = \begin{cases} 0, & \text{if } \mu \neq 0 \\ -x_1, & \text{if } \mu = 0 \end{cases}, \quad a_2 = \begin{cases} -\frac{1}{\mu}, & \text{if } \mu \neq 0 \\ 0, & \text{if } \mu = 0 \end{cases}$$

and

$$\tilde{b}_k = \begin{cases} \frac{(-1)^k}{\mu^{k+1}}, & \text{if } \mu \neq 0 \\ 0, & \text{if } \mu = 0 \end{cases}, \quad \text{for} \quad k \geqslant 0, \quad \tilde{a}_1 = 0, \quad \tilde{a}_2 = \begin{cases} 0, & \text{if } \mu \neq 0 \\ 1, & \text{if } \mu = 0 \end{cases}$$

are different from coefficients given in Case 1. Therefore

$$X^{\Delta_f} = \begin{cases} -\frac{1}{\mu}\frac{\partial}{\partial x_2} + \sum_{k \geqslant 0} \frac{(-1)^k}{\mu^{k+2}}\frac{\partial}{\partial u^{[k]}}, & \text{if } \mu \neq 0 \\ -x_1\frac{\partial}{\partial x_1}, & \text{if } \mu = 0 \end{cases} \tag{25}$$

and

$$X^{\sigma_f} = \begin{cases} \sum_{k \geqslant 0} \frac{(-1)^k}{\mu^{k+1}}\frac{\partial}{\partial u^{[k]}}, & \text{if } \mu \neq 0 \\ \frac{\partial}{\partial x_2}, & \text{if } \mu = 0. \end{cases} \tag{26}$$

Note that for $\mu = 0$ the vector fields X^{Δ_f} and X^{σ_f} coincide with (23).

Case 3 ($z = x_1$). A third possibility is to choose $z = x_1$ and define inversive closure $\mathscr{K}^*$ as a field of meromorphic functions in a finite number of variables x_1, x_2, $u^{[k]}$, $x_1^{\langle -\ell\rangle}$, $k \geqslant 0$, $\ell \geqslant 1$, where $x_1^{\langle -1\rangle} = \sigma_f^{-1}(x_1)$ and $x_1^{\langle -\ell\rangle} = \sigma_f^{-1}(x_1^{\langle -\ell+1\rangle})$.

Since X^{Δ_f} has the form (11), $\sigma_f(\mathrm{d}x_i) = \mathrm{d}x_i^\sigma$ and $\Delta_f(\mathrm{d}x_i) = \mathrm{d}x_i^\Delta$, $i = 1,2$, taking $\langle X, \mathrm{d}x_1\rangle^{\Delta_f}$, $\langle X, \mathrm{d}x_2\rangle^{\Delta_f}$, $\langle X, \mathrm{d}u^{[k]}\rangle^{\Delta_f}$, $k \geqslant 0$, we get equations (13), (14), (15), respectively. For $\langle X, x_1^{\langle -1\rangle}\rangle^{\Delta_f}$ we have

$$\langle X, \mathrm{d}x_1^{\langle -1\rangle}\rangle^{\Delta_f} = \quad 0 \quad = \langle X^{\Delta_f}, \mathrm{d}x_1\rangle + \left\langle X, \mathrm{d}\left(\frac{x_1 - x_1^{\langle -1\rangle}}{\mu}\right)\right\rangle = a_1\,,$$

which is different from (16) and (21) given for the vector field X and the differential of new variable $u^{\langle -1\rangle}$ in Case 1 and $x_2^{\langle -1\rangle}$ in Case 2, but taking $\langle X, \mathrm{d}x_1^{\langle -\ell-1\rangle}\rangle^{\Delta_f}$, $\ell \geqslant 1$, we get equations (17), i.e. $c_\ell = 0$, $\ell \geqslant 1$. Similarly, since X^{σ_f} have the form (12) and $\sigma_f(\mathrm{d}x_i) = \mathrm{d}x_i^\sigma$, $i = 1,2$, for $\langle X, \mathrm{d}x_1\rangle^{\sigma_f}$, $\langle X, \mathrm{d}x_2\rangle^{\sigma_f}$, $\langle X, \mathrm{d}u^{[k]}\rangle^{\sigma_f}$, $k \geqslant 0$, we have equations (18), (19), (20), respectively. For $\langle X, \mathrm{d}x_1^{\langle -1\rangle}\rangle^{\sigma_f}$ we have

$$\langle X, \mathrm{d}x_1^{\langle -1\rangle}\rangle^{\sigma_f} = \quad 0 \quad = \langle X^{\sigma_f}, \mathrm{d}x_1\rangle = \tilde{a}_1\,,$$

which is again different from (21) and (24) in Cases 1 and 2, but taking $\langle X, \mathrm{d}x_1^{\langle -\ell-1\rangle}\rangle^{\sigma_f}$, $\ell \geqslant 1$, we get equations (22), i.e. $\tilde{c}_\ell = 0$, $\ell \geqslant 1$. Then the coefficients c_ℓ, $\ell \geqslant 1$, b_k, $k \geqslant 0$, a_1, a_2, and $\tilde{c}_\ell$, $\ell \geqslant 1$, $\tilde{b}_k$, $k \geqslant 0$, $\tilde{a}_1$, $\tilde{a}_2$ are the same as in Case 2, so X^{Δ_f} and X^{σ_f} have the form (25) and (26), respectively. Moreover, for $\mu = 0$, X^{Δ_f} and X^{σ_f} coincide with (23). The fact that for $\mu = 0$ the vector fields X^{Δ_f} and X^{σ_f} are the same in all considered cases is related to the fact that $\sigma_f = \mathrm{id}$ for $\mu = 0$ and consequently, $\mathscr{K} = \mathscr{K}^*$.

Note that even if the vector field X is given by the finite linear combination (5), as in Example 3.2 where we have $X = \frac{\partial}{\partial x_2}$, it may happen that X^{σ_f} and X^{Δ_f} are the infinite vector fields as in Cases 2 and 3 of Example 3.2.

4. *p*-FORMS

In this section we unify the calculus of p-forms, extend the operators Δ_f and σ_f to p-forms, and prove some of their properties.

For any integer p, $p \geqslant 1$, consider the infinite set of symbols

$$\wedge^p \mathrm{d}\mathscr{C}^* = \{\mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 \wedge \cdots \wedge \mathrm{d}\zeta_p,\ \ \zeta_i \in \mathscr{C}^*,\ i = 1,\ldots,p\}$$

and denote by $\wedge^p\mathscr{E}$ the vector space spanned over $\mathscr{K}^*$ by the elements of $\wedge^p \mathrm{d}\mathscr{C}^*$:

$$\wedge^p\mathscr{E} := \mathrm{span}_{\mathscr{K}^*}\{\wedge^p \mathrm{d}\mathscr{C}^*\}$$

and

$$\wedge^0\mathscr{E} := \mathscr{K}^*\,.$$

In $\wedge^p\mathscr{E}$, $p \geqslant 2$, we consider the equivalence relation $\mathfrak{R}$ defined by the equalities

$$\mathrm{d}\zeta_{i_1} \wedge \mathrm{d}\zeta_{i_2} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} = (-1)^k \mathrm{d}\zeta_{j_1} \wedge \mathrm{d}\zeta_{j_2} \wedge \cdots \wedge \mathrm{d}\zeta_{j_p}, \tag{27}$$

where $\{i_1, i_2, \ldots, i_p\} = \{j_1, j_2, \ldots, j_p\}$ and k is the signature of the permutation $\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}$. The vector space $\wedge^p\mathscr{E}$ mod $\mathfrak{R}$ will be denoted[2] by $\mathscr{E}^p$. Its elements are called *forms of degree p* or simply *p-forms*. Every p-form $\alpha \in \mathscr{E}^p$ has a unique representative of the form

$$\alpha = \sum_{i_1 < \ldots < i_p} A_{i_1 \ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p}\,,$$

[2] Wedge of p one-forms.

where $A_{i_1\dots i_p} \in \mathscr{K}^*$. Usually such a representative will be used. For instance, if $[A_{12}\mathrm{d}x_1 \wedge \mathrm{d}x_2] = \{A_{12}\mathrm{d}x_1 \wedge \mathrm{d}x_2, -A_{12}\mathrm{d}x_2 \wedge \mathrm{d}x_1\} \in \mathscr{E}^2$, then $A_{12}\mathrm{d}x_1 \wedge \mathrm{d}x_2$ is chosen as a representative of the considered 2-form. By the constructions described above we obtain a sequence of vector spaces $\mathscr{E}^0 := \mathscr{K}^*$, $\mathscr{E}^1 := \mathscr{E}$, $\mathscr{E}^2, \mathscr{E}^3, \dots, \mathscr{E}^p, \dots$

The *exterior product* (alternatively called the *wedge product*) of a p-form representative $\omega_1 = \sum_{i=1}^k F_i \mathrm{d}\zeta_{i_1} \wedge \dots \wedge \mathrm{d}\zeta_{i_p}$ and a q-form representative $\omega_2 = \sum_{j=1}^{\ell} G_j \mathrm{d}\zeta_{j_1} \wedge \dots \wedge \mathrm{d}\zeta_{j_q}$, denoted as $\omega_1 \wedge \omega_2$, is defined by a $(p+q)$-form representative in the following way:

$$\omega_1 \wedge \omega_2 = \sum_{i=1}^{k} \sum_{j=1}^{\ell} F_i G_j \mathrm{d}\zeta_{i_1} \wedge \dots \wedge \mathrm{d}\zeta_{i_p} \wedge \mathrm{d}\zeta_{j_1} \wedge \dots \wedge \mathrm{d}\zeta_{j_q},$$

where $F_i, G_j \in \mathscr{K}^*$ and $\zeta_{i_l}, \zeta_{j_s} \in \mathscr{C}^*$, $l = 1, \dots, p$, $s = 1, \dots, q$. This definition does not depend on the choice of the representative in the equivalence class. It can be easily verified that the exterior product is bilinear and associative, moreover, it induces a map $\wedge : \mathscr{E}^p \times \mathscr{E}^q \to \mathscr{E}^{p+q}$, $p, q \geqslant 0$, given by $\wedge(\omega_1, \omega_2) = \omega_1 \wedge \omega_2$ for some representatives ω_1 and ω_2 in equivalence classes. In general, the exterior product for representatives ω_1 and ω_2 is not commutative, since (27) implies

$$\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1.$$

Note that for $F, G \in \mathscr{E}^0$ and $\zeta_i \in \mathscr{C}^*$ we have $F \wedge G = F \cdot G \in \mathscr{E}^0$ and

$$F \wedge \mathrm{d}\zeta_1 \wedge \dots \wedge \mathrm{d}\zeta_p = F \mathrm{d}\zeta_1 \wedge \dots \wedge \mathrm{d}\zeta_p \in \mathscr{E}^p.$$

Exterior differential d is an $\mathbb{R}$-linear operator

$$\mathrm{d} : \mathscr{E}^p \to \mathscr{E}^{p+1}$$

that satisfies the following properties:
(i) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^s \alpha \wedge \mathrm{d}\beta$, where $\alpha \in \mathscr{E}^s$ and $\beta \in \mathscr{E}^{p-s}$,
(ii) if $F \in \mathscr{K}^*$, then $\mathrm{d}F$ coincides with ordinary differential (see (3)),
(iii) $\mathrm{d}^2 = 0$, where $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} : \mathscr{E}^p \to \mathscr{E}^{p+2}$.

Properties (i)–(iii) define uniquely the operator d.

A differential p-form $\alpha \in \mathscr{E}^p$ is said to be *closed* if $\mathrm{d}\alpha = 0$ and *exact* if there exists a differential $(p-1)$-form $\beta \in \mathscr{E}^{p-1}$ such that $\alpha = \mathrm{d}\beta$. An exact differential form is closed.

A subspace $\mathscr{V} \subset \mathscr{E}^1$ is said to be *closed* (or completely *integrable*) if $\mathscr{V}$ admits (locally) a basis composed of closed forms. To check whether the subspace $\mathscr{V}$ is integrable, one may use the Frobenius Theorem (see for instance [15]).

Theorem 4.1 (Frobenius)**.** *Let $\mathscr{V}$ be the subspace of $\mathscr{E}^1$ generated by the one-forms $\{\omega_1, \dots, \omega_r\}$. $\mathscr{V}$ is closed if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_r = 0,$$

for any $i = 1, \dots, r$.

Example 4.2. Let $x \neq 0$. The one-form $\omega = x\mathrm{d}u - u\mathrm{d}x$ is not closed (therefore neither exact) since $\mathrm{d}\omega = 2\mathrm{d}x \wedge \mathrm{d}u$. However, the vector space $\mathrm{span}_{\mathscr{K}^*}\{\omega\}$ is integrable since $\mathrm{d}\omega \wedge \omega = 0$ and one may choose the integrating factor $A = \frac{1}{x^2}$ such that $A\omega = \mathrm{d}\left[\frac{u}{x}\right]$.

Let

$$\mathbb{E} := \mathscr{E}^0 \oplus \mathscr{E}^1 \oplus \cdots$$

An element $\omega \in \mathbb{E}$ can be written in a unique way as

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^N$$

for some $N \geqslant 0$, where $\omega^p \in \mathscr{E}^p$ is called the pth component of ω. $\mathbb{E}$ is called the *exterior algebra over* $\mathscr{E}$. It has a structure of a graded algebra with multiplication given by the exterior product $\wedge$.

Note that the space of forms with the exterior product is a ring. A subring $I \subset \mathbb{E}$ is called an *algebraic ideal of* $\mathbb{E}$ if

(i) $\alpha \in I$ implies $\alpha \wedge \beta \in I$ for all $\beta \in \mathbb{E}$,

(ii) $\alpha \in I$ implies that all of its components in $\mathbb{E}$ are contained in I.

Observe that $\alpha \in I$ implies $\beta \wedge \alpha \in I$ for all $\beta \in \mathbb{E}$. Thus algebraic ideals are two-sided ideals. An *exterior differential system* is an algebraic ideal $\mathscr{I}$ that is stable with respect to exterior differentiation.

The operators $\Delta_f : \mathscr{K}^* \to \mathscr{K}^*$ and $\sigma_f : \mathscr{K}^* \to \mathscr{K}^*$, related to system (1), induce the operators $\Delta_f : \mathscr{E}^p \to \mathscr{E}^p$ and $\sigma_f : \mathscr{E}^p \to \mathscr{E}^p$ by

$$\begin{aligned}\Delta_f \left(\sum_{i_1<\ldots<i_p} A_{i_1\ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \right) := \sum_{i_1<\ldots<i_p} \Big[& A^{\Delta_f}_{i_1\ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\Delta_f}_{i_1} \wedge \mathrm{d}\zeta_{i_2} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \\ & + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \mathrm{d}\zeta^{\Delta_f}_{i_2} \wedge \mathrm{d}\zeta_{i_3} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} + \cdots \\ & + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta^{\sigma_f}_{i_{p-1}} \wedge \mathrm{d}\zeta^{\Delta_f}_{i_p} \Big]\end{aligned} \tag{28}$$

and

$$\sigma_f \left(\sum_{i_1<\ldots<i_p} A_{i_1\ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \right) := \sum_{i_1<\ldots<i_p} \left[A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta^{\sigma_f}_{i_p} \right], \tag{29}$$

where $\zeta_{i_1}, \ldots, \zeta_{i_p} \in \mathscr{C}^*$ and $A_{i_1\ldots i_p} \in \mathscr{K}^*$.

Proposition 4.3. *Let $\omega \in \mathscr{E}^p$, $p \geqslant 1$. Then for homogeneous time scale $\mathbb{T}$*

$$\mathrm{d}\left[\omega^{\Delta_f}\right] = [\mathrm{d}\omega]^{\Delta_f} \text{ and } \mathrm{d}\left[\omega^{\sigma_f}\right] = [\mathrm{d}\omega]^{\sigma_f} .$$

Proof. Let $\omega = \sum_{i_1<\ldots<i_p} A_{i_1\ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \in \mathscr{E}^p$. Then we have

$$\mathrm{d}\omega = \sum_{i_1<\ldots<i_p} \mathrm{d}A_{i_1\ldots i_p} \wedge \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p},$$

$$\begin{aligned}\omega^{\Delta_f} = \sum_{i_1<\ldots<i_p} \big[& A^{\Delta_f}_{i_1\ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\Delta_f}_{i_1} \wedge \mathrm{d}\zeta_{i_2} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \\ & + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \mathrm{d}\zeta^{\Delta_f}_{i_2} \wedge \mathrm{d}\zeta_{i_3} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} + \cdots + A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta^{\sigma_f}_{i_{p-1}} \wedge \mathrm{d}\zeta^{\Delta_f}_{i_p} \big]\end{aligned}$$

and

$$\omega^{\sigma_f} = \sum_{i_1<\ldots<i_p} A^{\sigma_f}_{i_1\ldots i_p} \mathrm{d}\zeta^{\sigma_f}_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta^{\sigma_f}_{i_p}.$$

Using (4) and definitions (28), (29), we have

$$\begin{aligned}
\mathrm{d}\left[\omega^{\Delta_f}\right] = & \sum_{i_1<\ldots<i_p} \Big[\mathrm{d}\left(A^{\Delta_f}_{i_1\ldots i_p}\right)\wedge \mathrm{d}\zeta_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} + \mathrm{d}\left(A^{\sigma_f}_{i_1\ldots i_p}\right)\wedge \mathrm{d}\zeta^{\Delta_f}_{i_1}\wedge \mathrm{d}\zeta_{i_2}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} \\
& + \mathrm{d}\left(A^{\sigma_f}_{i_1\ldots i_p}\right)\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge \mathrm{d}\zeta^{\Delta_f}_{i_2}\wedge \mathrm{d}\zeta_{i_3}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} + \cdots + \mathrm{d}\left(A^{\sigma_f}_{i_1\ldots i_p}\right)\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta^{\sigma_f}_{i_{p-1}}\wedge \mathrm{d}\zeta^{\Delta_f}_{i_p}\Big] \\
= & \sum_{i_1<\ldots<i_p} \Big[\left(\mathrm{d}A_{i_1\ldots i_p}\right)^{\Delta_f}\wedge \mathrm{d}\zeta_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} + \left(\mathrm{d}A_{i_1\ldots i_p}\right)^{\sigma_f}\wedge \mathrm{d}\zeta^{\Delta_f}_{i_1}\wedge \mathrm{d}\zeta_{i_2}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} \\
& + \left(\mathrm{d}A_{i_1\ldots i_p}\right)^{\sigma_f}\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge \mathrm{d}\zeta^{\Delta_f}_{i_2}\wedge \mathrm{d}\zeta_{i_3}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p} + \cdots + \left(\mathrm{d}A_{i_1\ldots i_p}\right)^{\sigma_f}\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta^{\sigma_f}_{i_{p-1}}\wedge \mathrm{d}\zeta^{\Delta_f}_{i_p}\Big] \\
= & \left[\sum_{i_1<\ldots<i_p} \mathrm{d}A_{i_1\ldots i_p}\wedge \mathrm{d}\zeta_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p}\right]^{\Delta_f} = [\mathrm{d}\omega]^{\Delta_f},
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\left[\omega^{\sigma_f}\right] = & \sum_{i_1<\ldots<i_p} \mathrm{d}\left(A^{\sigma_f}_{i_1\ldots i_p}\right)\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta^{\sigma_f}_{i_p} = \sum_{i_1<\ldots<i_p} \left(\mathrm{d}A_{i_1\ldots i_p}\right)^{\sigma_f}\wedge \mathrm{d}\zeta^{\sigma_f}_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta^{\sigma_f}_{i_p} \\
= & \left[\sum_{i_1<\ldots<i_p} \mathrm{d}A_{i_1\ldots i_p}\wedge \mathrm{d}\zeta_{i_1}\wedge\cdots\wedge \mathrm{d}\zeta_{i_p}\right]^{\sigma_f} = [\mathrm{d}\omega]^{\sigma_f}. \qquad \square
\end{aligned}$$

Now let us consider a $\mathscr{K}^*$-linear operator $i_X : \mathscr{E}^p \to \mathscr{E}^{p-1}$ associated with a vector field $X \in \mathscr{E}'$ that satisfies the following properties:
1. $i_X(\omega_1 \wedge \omega_2) = i_X(\omega_1)\wedge\omega_2 + (-1)^p\omega_1 \wedge i_X(\omega_2)$, where p is the degree of ω_1;
2. $i_X(F) = 0$, for all $F \in \mathscr{K}^*$;
3. $i_X(\mathrm{d}x_i) = X_i$, where X_i denotes the ith component of X.

Note that $i_X(\mathrm{d}\zeta_i) = \langle X, \mathrm{d}\zeta_i\rangle$ so that for a differential 2-form $\vartheta = \sum_{i,j} a_{ij}\mathrm{d}\zeta_i \wedge \mathrm{d}\zeta_j$ we have

$$i_X\vartheta = \sum_{i,j} a_{ij}\left(\langle X, \mathrm{d}\zeta_i\rangle \mathrm{d}\zeta_j - \langle X, \mathrm{d}\zeta_j\rangle \mathrm{d}\zeta_i\right).$$

Taking the delta derivative of both sides in this identity yields, by (7),

$$[i_X\vartheta]^{\Delta_f} = i_{X^{\Delta_f}}\vartheta^{\sigma_f} + i_X\vartheta^{\Delta_f}.$$

The *characteristic vector fields* associated with an exterior differential system $\mathscr{I}$ are the elements of the set

$$A(\mathscr{I}) = \left\{X \in \mathscr{E}' \mid i_X(\mathscr{I}) \subset \mathscr{I}\right\}.$$

The annihilator $C(\mathscr{I})$ of $A(\mathscr{I})$ is the *characteristic system of $\mathscr{I}$*. The characteristic system is completely integrable, see [15].

5. CONCLUSIONS

The paper extends further the algebraic formalism of differential one-forms, described in [5] for nonlinear control systems defined on homogeneous time scale. First, we unify the calculus of p-forms by extending the main concept of time scale calculus – delta-derivative – to p-forms and prove a number of its properties. In particular, we prove that the operators of the exterior derivative and the delta derivative commute when applied to p-forms. Second, we introduce the dual space of the vector fields over the field of meromorphic functions. The vector fields may be interpreted as the linear mappings from the space of one-forms into the field of meromorphic functions. A collection of *Mathematica* functions has been developed in order to simplify the computations with vector fields. These functions are part of the larger package NLControl, addressing various nonlinear control problems. Moreover, the functions are made available on the NLControl website [16], so that everyone can use them via the internet browser.

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Diferentsiaalsete p-vormide ja vektorväljade algebraline formalism mittelineaarsete juhtimissüsteemide jaoks ajaskaaladel

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Varasemas artiklis on välja töötatud diferentsiaalsetel üksvormidel põhinev algebraline formalism mittelineaarsete juhtimissüsteemide jaoks homogeensetel ajaskaaladel, mis võimaldab ühildada pidevate ja diskreetsete süsteemide uurimist. Antud artiklis on algebralist formalismi edasi arendatud: on ühildatud ka p-vormide arvutus ja defineeritud (üksvormide ruumi) duaalne ruum üle meromorfsete funktsioonide korpuse, mille elementideks on vektorväljad. Ajaskaala põhimõiste, delta-tuletis, on üldistatud nii p-vormidele kui ka vektorväljadele. On tõestatud antud operaatori rida omadusi, muuhulgas p-vormi delta-tuletise kommuteeruvus diferentsiaali võtmise operatsiooniga.