TRAMES, 2006, **10 (60 / 55)**, 1, 57 – 80

SHEAF-THEORETIC FORMAL SEMANTICS

Oleg Prosorov

St. Petersburg Department of V. A. Steklov Institute of Mathematics

Abstract. We outline a sheaf-theoretical framework for a discourse interpretation theory developed in our previous works on *formal semantics*. This theory applies rigorous mathematical methods in studying the process of interpretation of a natural language text written with good grace and intended for a human understanding. We propose the generalization of Frege's principle of compositionality of meaning which extends its domain from the level of isolated sentence to that of a whole text and takes into account the multiplicity of senses and meanings of words, sentences and texts. The present sheaf-theoretical formal semantics provides a mathematical model of the text interpretation process while rejecting the attempts to codify interpretative practice as a kind of calculus.

Keywords: formal semantics, phonocentric topology, fragmentary meaning, hermeneutic circle, sheaf, bundle, étale bundle, Frege's generalized compositionality principle, Frege's generalized contextuality principle, category, functor, Frege duality, textual space

1. Introduction

We consider some unspecified Indo-European language, e.g., English, French, German, Russian, as a means of communication. We deal mostly with a written type of linguistic communication and so its basic units are discourses or texts. All the texts we consider are supposed to be written with good grace and intended for human understanding; we call *admissible* the texts of this kind. All sequences of words written in order to confuse a reader or to imitate some human writings are cast aside as irrelevant to linguistic communication.

The classic approaches to semantics of natural language are based on the implicit premise that any language is nothing more than the set of all its correct sentences. These approaches are very restrictive and yet inadequate to everyday human practice of language communication. When a person wants to express his thoughts to somebody, he needs to utter some discourse or to write some text, and to understand this data is quite another thing than to understand the set of all sentences it was made up. This is why the semantics of natural language should be defined as a discipline studying the discourse and text understanding. To go further, we need to define rigorously what a text is in our formalism. Clearly any text is not just a set of its sentences as the sentence is not a set of its words. What is important is the order they ought to be read. In addition, the same words may occur in several places of one sentence and the same sentences may occur in several places of one text. So from a mathematical point of view, we ought to consider a sentence as a sequence of its words and a text as a sequence of its sentences. Likewise any part of a considered whole is simply a subsequence of a given sequence. Any mathematical structure on a given text, such as topology, sheaves etc., is supposed to be defined on the *functional graph* of a corresponding sequence. Henceforth, we shall simply identify a given text with the graph of its corresponding sequence.

2. Basic concepts

In the formal analysis of text understanding, we distinguish the semantic notions *sense*, *meaning* and *reference*. This triad of concepts formalizes a certain distinction that seems to appear in various forms all over the history of language studies. To avoid the possible misunderstanding from the very beginning, we would like to make precise our usage of these key terms and to point out that our distinction *sense/meaning* differs from Frege's classic *Sinn/Bedeutung* distinction, whereas we accept *reference* to be an English translation of Frege's *Bedeutung*. Our aim is not to propose some competitive alternatives to Frege's *Sinn/Bedeutung* distinction but to find some adequate semantic concepts pertinent as instruments for the rigorous formal analysis of the text interpretation process where Frege's classic compositionality and contextuality principles are involved. However, one can find our distinction *sense/meaning* in the different usage of the word 'Sinn' in early writings by Frege before he had formalized the *Sinn/Bedeutung* distinction in his classic work of 1892. We consider *sense* and *meaning* as being the basic notions to be expressed by means of examples and descriptions, and instead of their analysis in terms of more basic ones, we seek for key mathematical structures that underlie the process of discourse or text understanding.

We accept the term *fragmentary meaning* of some fragment of a given text to be the content which is grasped when the reader has understood this fragment in some particular situation of reading. But it depends on so many factors such as the personality of the reader, the situation of the reading, many kinds of presuppositions and prejudices summed up in the reader's attitude, etc., which we call *sense* or *mode of reading*; every reading is only an interpretation where the historicity of the reader and the historicity of the text are involved; thus in our usage, a *fragmentary meaning* is immanent not in a given fragment, but in an interpretative process of its reading. In our approach, the notion *sense* (or *mode of reading*) may be considered as a secular remake of the exegetical approach to this notion in the medieval theology. The Fathers of the Church have distinguished the four senses of Sacred Scripture: "Littera gesta docet, quid credas allegoria, moralis quid agas, quo tendas anagogia". In other words, our approach defines the term *sense* as a kind of semantic orientation in the interpretative process which relates to the totality of the message to understand, as some *mode of reading*. At the level of text, it may be *literal, allegoric, moral, eschatological, naïve, psychoanalytical*, etc. At the level of sentence, it may be *literal* or *metaphoric* (*indirect*). At the level of word, it may be *literal* or *figurative*.

In our approach, the reader grasps a *fragmentary meaning* in a particular interpretative process guided by some *mode of reading* or *sense* adopted in accordance with his attitude and based on the linguistic competence, which is rooted in the social practice of communication with others using the medium of language. Note that, following this terminology, we can read one and the same text in many different senses (moral, historical, etc.) to realize, in result, that we have grasped the different meanings. Likewise for a sentence or expression.

It seems that the usage of the key terms *sense, meaning* is in accordance with their everyday usage as common English words (likewise for the French terms *sens* and *signification*). As for the term *sense*, it should be mentioned that in French the word 'sens' literally equals 'direction' and as figurative it may be *littéral*, *strict*, *large*, *naïf*, *bon*, *platonicien*, *leibnitzéen*, *frégéen*, *kripkéen*, etc. In English, in figurative usage, sense may also be *literal*, *narrow*, *wide*, *naïve*, *common*, *Platonic*, *Leibnizian*, *Fregean*, *Kripkean*, etc. In this usage, the term *sense* deals with the totality of discourse, text, expression or word and involves our subjective premises that what is to be understood constitutes a meaningful whole. In this usage, the term *sense* or *mode of reading* concerns the reader's interest in the subject matter of the text; it is a kind of questioning that allows a reader to enter into a dialogue with the author. So our usage of the term *sense* as a mode of reading is near to that proposed by the exegetic concept of the four senses of the Sacred Scripture, whereas our usage of the term *fragmentary meaning* as the content grasped in some particular situation of reading corresponds rather to the common usage of ordinary English words.

We would like to stress here the difference between this acceptance and the Fregean acceptance of *Sinn* as the "mode of presentation of reference" which is often illustrated by the famous example of "morning star" and "evening star". We consider it as an example of two different texts or expressions; each of them may be interpreted in many different *senses* or *modes of reading* and, following a chosen sense (mode of reading), we can grasp the different meanings of it.

In accordance with our acceptance, we consider a meaning of a given expression which is grasped in some particular situation of communication (real or fictive, direct or mediated) to be rather what the reader or listener understands as a response to an implicit question. One finds such a meaning by asking himself: "What does it mean in this or that sense?" or ("Qu'est-ce que cela signifie dans tel ou tel sens ?"). Words, expressions, texts mean what the members of a given linguistic community at a given time understand them to mean. The later Wittgenstein expressed this point of view in his famous slogan: "The meaning is use". So it is an odd misconception to think that a word (an expression, a text) has only one true meaning. Being composed during the interpretative process, some particular meaning s of a fragment $U \subset X$ is rooted in the *use* and is motivated by the adopted *mode of reading* $\mathscr{F}$.

But the fragmentary meaning should not be understood as some mental state of the reader because the mental states of two readers could neither be identified, nor compared in any reasonable way; on the contrary, our approach is based on the criterion of equality between the fragmentary meanings we formulate explicitly. In our usage, the term *fragmentary meaning* should not be understood in the Tarski/Montague style as the relation between words and world; nor should it be related to any kind of truth-value or truth-conditions because the understanding of, e.g., novels is achieved regardless of any assumption about verifiability.

The understanding of meaning and the knowledge of truth both relate to the world, but in different ways. We observe that a meaning s of some fragment U of a given text X is understood by the reader as an objective result of interpretation of this passage U; its 'objectivity' carries no claim of correspondence to reality, but is grounded in the conviction that this meaning s may be discussed with anybody in some kind of dialogue (actual or imaginary) where such a meaning s may be finally shared by the participants or may be compared with any other meaning t of the same fragment U. We formulate the criterion for such a comparison procedure as some definition of equality (S). This kind of objectivity of meaning is based not only on the shared language, but principally on the shared experience as a common life-world and it deals so with the reality. According to Gadamer, this being-with-each-other is a general building principle both in life and in language. The understanding results from being together in a common world. This understanding as a presumed agreement on 'what this fragment U wants to say' becomes for the reader its meaning s. In this usage, the meaning of an expression is the content that the reader grasps when he understands it; and this can be done regardless of the ontological status of its reference. The process of coming to some fragmentary meaning s of a fragment U may be thought of as an exercise of the human capacity of naming and understanding; it is a fundamental characteristic of human linguistic behavior.

3. Phonocentric topology

The reading of text as well as the utterance of discourse is always a process that develops in time, and so it inherits in some way its order structure. From a linguistic point of view, this order structure is known as a notion of *linearity* or that of *word order*. At the level of text, it is a natural linear order $\leq$ of sentences reading the text bears on. It is well-known that any order structure carries several standard topological structures (Erné 1991) but it is not a question to graft some topology onto a given text. We argue that any admissible text has an underlying topological structure which arises quite naturally.

In the process of reading, the understanding is not postponed until the final sentence of a given text. So the text should have the meaningful parts and the meanings of these parts determine the meaning of the whole, as it is claimed by the principle of *hermeneutic circle*. A central task of any semantic theory is to explain how these local understandings of the constitutive meaningful parts produce the global understanding of the whole. Whereas a description of some mathematical structure in terms of these constitutive meaningful parts may be treated as a kind of syntactic theory concerned with a considered semantic level. The philological investigations are abound in examples of meaningful fragments quoted from the studied texts. Thus a meaningful part might be a subject of comment or discussion for being considered as worth interpretation. Certainly, not each subsequence of a given text is meaningful, but a meaningful fragment becomes understood in the process of reading and rereading. It seems to be quite in agreement with our linguistic intuition that:

(i) *an arbitrary union of meaningful parts of an admissible text is meaningful;*

(ii) *a non-empty intersection of two meaningful parts of an admissible text is meaningful.*

For an admissible text is supposed to be meaningful as a whole by definition, it remains only to define the meaning of its empty part (e.g., as a one-element set) in order to provide it with some *topology* in a strict mathematical sense, where the open sets are all the meaningful parts. Thus any admissible text may be endowed with a *semantic topology* where the open sets are defined to be all its meaningful parts (Prosorov 2004). In the following, we often use the term *fragment* as equivalent to that of *open subset* in the case of topological space related to text. Now, however, the question arises, what formal criteria would be given for the meaningfulness of a part $U \subset X$? The concepts of *fragmentary meaning* and *meaningful fragment* are closely related, for they should come together in the matter of natural language text understanding. We have noted at the very beginning, that our theory concernes only the texts referred to as *admissible*, which are supposed to be written for a human understanding. So the meaningful parts are supposed to be those which are intended to convey the communicative content. Therefore, an admissible text should respect good order and arrangement, as each part ought to fall into its right place; so the ordinary reading process inherits a natural temporality of phonetic phenomena, we call this kind of semantic topology as *phonocentric*.

The natural process of reading (from right to left and from top to bottom) supposes that understanding of any sentence x of the text X should be achieved on the basis of the text's part already read, because the interpretation cannot be postponed, although it may be made more precise and corrected in further reading and rereading. This is a fundamental feature of a competent reader's linguistic behavior. Rastier describes it as follows:

> *Alors que le régime herméneutique des langages formels est celui du suspens, car leur interprétation peut se déployer après le calcul, les textes ne connaissent jamais le suspens de l'interprétation. Elle est compulsive et incoercible. Par exemple, les mots inconnus, les noms propres, voire les non-mots sont interprétés, validement ou non, peu importe.* (Rastier 1995: 165–166)

Thus for every pair of distinct sentences x, y of an admissible text X, there exists an open (i.e., meaningful) part of X that contains one of them (to be read first in the natural order $\leq$ of sentences) and doesn't contain the other. Hence the admissible text endowed with the phonocentric topology should satisfy the *separation axiom* T_0 of Kolmogoroff and so it is a T_0-*space*. This characteristic might be posed as a formal definition distinguishing the phonocentric topology between the other semantic topologies. According to our conceptual distinction *sense/meaning*, we consider sense as a kind of semantic orientation in the interpretative process which relates to the totality of message to understand. Thus we suppose that any part $U \subset X$ which is meaningful in one sense (or mode of reading) should remain meaningful under the passage to some another sense in the ordinary process of reading. It should be noticed that another concept of meaning or criteria of meaningfulness would imply another definition of meaningful fragments and so will define yet another type of *semantic topology*. Thus, for the class of scientific texts, it will be pertinent the criterion of *verifiability* or, following another philosophical approach, the criterion of *falsifiability*; one may also adopt, following Einstein, some *operational* criterion of meaningfulness. In computer science, there is also an *operational semantics* as a way to assigne meaning to computer programs, which are texts in a formal language.

Note that it is a very difficult philosophical problem to elaborate any definition of meaningfulness. Essential for our work, however, is the simple observation that any reasonable definition of meaning or criterion of meaningfulness gives rise to certain topology on the discourse domain. This allows to interpret several tasks of discourse analysis in topological terms. It seems also to be useful in matter of comparing different notions of meaning: one may pose that the notion τ_1 of meaning is stronger than another one, say τ_2, if the topology defined by τ_1 is stronger than the topology defined by τ_2, that is if the identity map of underlying sets gives rise to the continuous map $(X, \tau_1) \to (X, \tau_2)$ of the topological spaces defined as above. From this point of view, the formal logic gives an example of the strongest semantic topology, i.e., discrete, because all the propositions are considered as being true or false and so meaningful, thus all the points are open. On the contrary, our analysis concernes a more vast domaine of admissible texts where not all sentences are true or false. In our acceptance, the meaning of an expression is the content which the reader or listener grasps when understands it; and this can be done regardless of the ontological status of its referent. So Frege writes in (1960: 63) that “the thought remains the same whether ‘Odysseus’ has reference or not”. According to one of Frege’s remarks to the work of Jourdain of 1912 on the history of logic, which is reproduced as footnote n° 6 in (Frege 1967: 11):

> *We must be able to express a thought without affirming that it is true. If we want to characterize a thought as false, we must first express it without affirming it, then negate it, and affirm as true the thought thus obtained. We cannot correctly express a hypothetical connection between thoughts at all if we cannot express thoughts without affirming them, for in the hypothetical connection neither the thought appearing as antecedent nor that appearing as consequent is affirmed.*

According to another his formulation[1] from the famous work *Über Sinn und Bedeutung*: “A judgement is not mere grasping of a thought, but the recognition of its truth”. So, following Frege, we can express a thought and we can grasp a thought without affirming or recognition of its truth. Likewise for any (admissible) text, we can understand its sentences and its fragments regardless of truth-values or truth-conditions.

Recall that a topological space X is an *Alexandroff space* (or *A-space*) if the set $\mathscr{O}(X)$ of all its open sets is closed under arbitrary intersections. For an admissible text being finite, it defines a finite space and thus it is an A-space. As we have mentioned above, not all the subsets of an admissible text are meaningful, and hence the phonocentric topology isn’t *discrete*. On the other hand, there are certainly the proper meaningful parts in an admissible text, hence the phonocentric topology isn’t *coarse*.

Let X be an admissible text. For a sentence $x \in X$, we define U_x to be the intersection of all the meaningful parts that contain x. In other words, for a given sentence x, the part U_x is a smallest open neighborhood of x. It is clear that $x \in U_y$ if and only if $y \in \mathrm{cl}(\{x\})$, where $\mathrm{cl}(\{x\})$ denotes the closure of the one-element set $\{x\}$. This relation ‘x is contained in all open sets that contain y’ is usually called a *specialization*, and some authors denote it as $y \preceq x$ (e.g., Erné 1991: 59) contrary to

[1] English translation by G. Sundholm (2002: 578).

others who denote it as $x \preceq y$ (e.g., May 2003: 2). As for the notation choice, we follow rather (May 2003) to define a relation $\preceq$ on the text X by setting $x \preceq y$ if and only if $x \in U_y$ or, equivalently, $U_x \subset U_y$.

Proposition. *The set of all open sets of the kind U_x is a basis of a phonocentric topology on X. The phonocentric topology on an admissible text defines a partial order structure $\preceq$ on it by means of specialization; the initial phonocentric topology can be recovered from this partial order $\preceq$ in a unique way.*

This is a linguistic variant of a well-known general result concerning the relations between topological and order structures on a finite set. That is, the category of finite T_0-topological spaces and continuous maps is isomorphic to the category of finite partially ordered sets (*posets*) and monotone maps (Erné 1991, May 2003). Namely, given a finite poset $(X, \leqslant)$, one defines a T_0-topology on X by means of the basis τ constituted of all sets $\{l \mid l \leqslant x\}$. Thus one obtain a functor $L\colon (X, \leqslant) \to (X, \tau)$ acting identically on the underlying set maps, which is a functor from the category of posets and monotone maps to the category of topological spaces and continuous maps. Conversely, the specialization functor Q, assigning to each finite space (X, τ) a poset $(X, \preceq)$ with the specialization order $\preceq$ and acting identically on the underlying set maps, is a functor from the category of topological spaces and continuous maps to the category of posets and monotone maps[2]. From a mathematical point of view, the study of one of these two categories is logically equivalent to the study of another.

These considerations may be repeated with a slight modifications in order to define a *phonocentric* topology at each semantic level of a given admissible text. At each semantic level (text, sentence, word), we distinguish its *primitive elements* which are the points of a corresponding topological space considered to be the *whole* at this level. The passage from one semantic level to another immediately superior consists in gluing of the whole space into a point of the higher level space.

Thus at each semantic level (text, sentence, word) of a given admissible text, there exist two natural order structures:

(i) *the specialization order $x \preceq y$ defined by applying the specialization functor Q to the natural phonocentric topology of a considered semantic level;*[3]

(ii) *the linear order $x \leq y$ of ordinary text reading.*

On the other hand, at each semantic level (text, sentence, word) of a given admissible text, there exist two topological structures:

(i) *the natural phonocentric topology of a considered semantic level;*

(ii) *the topology defined by applying the functor L to the linear order $x \leq y$ of ordinary text reading.*

---

[2] These notations L, Q of (Erné 1991: 60) originate from the words "low" and"quasiorder".

[3] At the level of sentence, one might call it as *structural order* (or *ordre structural* in French) following the terminology of Tesnière (1959) but this terminology, though suggestive, seems to be unsuitable here because it gives rise to some embarrassing expressions like, e.g., "structural order structure".

Note that, in these notations at the semantic level of text, the relation $x \preceq y$ implies the relation $x \leq y$ for all $x, y \in X$. Thus the identity map of underlying sets gives rise to the continuous map $L(X, \preceq) \to L(X, \leq)$ of corresponding topological spaces. So the necessary linearization during the writing process results in weakening of topology by the transition from $L(X, \preceq)$ to $L(X, \leq)$. The process of interpretation consists in backward recovering of the phonocentric topology on a given text.

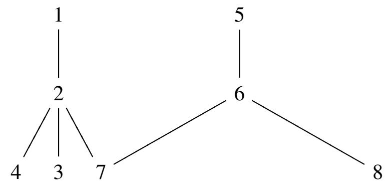
As soon as we have defined a phonocentric topology, we may seek to interpret some linguistic notion (at the concerned semantic level) in topological terms and then to study it by topological means. Take for example a property of a literary work to be the communicative unity of meaning. So for any two novels X and Y yet of the same kind, say historical, detective or biographical, their concatenation Z under one and the same cover doesn't constitute a new one. What does it mean, topologically speaking? We see that for any $x \in X$ there exists an open neighborhood U of x that doesn't meet Y, and for any $y \in Y$ there exists an open neighborhood V of y that doesn't meet X. Thus $Z = X \bigsqcup Y$, i.e., Z is a disjoint union of two non-empty open subsets X and Y; hence Z isn't connected. Recall that a space X is said to be *connected* if it is not the disjoint union of two non-empty open subsets. It is clear that each minimal basic open set U_x is connected.

In the mathematical order theory, there exists a simple intuitive tool for the graphical representation of a finite poset, called Hasse diagram (Stanley 1986). For a poset $(X, \preceq)$, the *cover relation* $\prec$ is defined by: '$x \prec y$ if and only if $x \preceq y$ and there exists no element $z \in X$ such that $x \preceq z \preceq y$'. In this case, we say that y *covers* x. For a given poset $(X, \preceq)$, its Hasse diagram is defined as the graph whose vertices are the elements of X and whose edges are those pairs $\{x, y\}$ for which $x \prec y$. In the picture, the vertices of Hasse diagram are labeled by the elements of X and the edge $\{x, y\}$ is drawn by an arrow going from x to y (or sometimes by an indirected line, but in this case y is displayed lower than x).

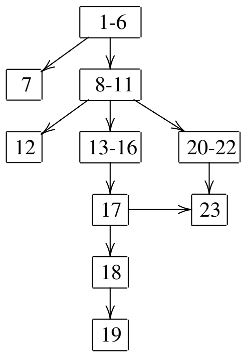
The usage of some kind of Hasse diagram under the name of Leitfaden is widely spread in the mathematical books to facilitate the understanding of logical dependence of the chapters. The poset considered in this usage is usually constituted not of all sentences but of all chapters of the book. So, in the introduction to (Serre 1979) is written: "The logical relations among the different chapters are made more precise in the *Leitfaden* below." and there is the following Hasse diagram:

$$
\begin{tikzcd}
 & 1 \arrow[d] & & & \\
 & 2 \arrow[d] & & & \\
 & 3 \arrow[d] & & 7 \arrow[dl] \arrow[dr] & \\
 & 4 \arrow[d] & 10 \arrow[dd] & & 8 \arrow[d] \\
 & 5 \arrow[dl] \arrow[dr] & & & 9 \arrow[d] \\
6 & & 12 \arrow[dr] & & 11 \arrow[dl] \\
 & & & 13 \arrow[dl] \arrow[dr] & \\
 & & 14 & & 15
\end{tikzcd}
$$

We cite yet another example of Hasse diagram from (Manin 1977), where it appears under the title of "Interdependence of Chapters":



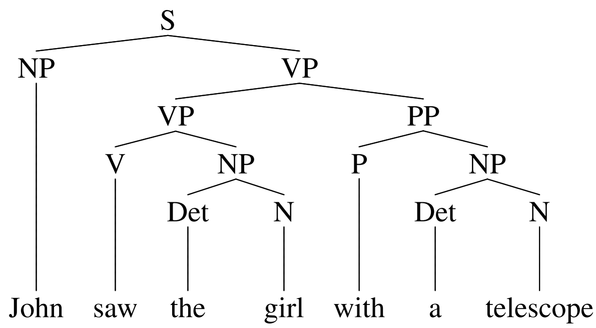
These two Leitfadens, as many other their examples, surely presuppose the linear reading of paragraphs within each chapter. Thus, they may be "split" in order to draw the corresponding Leitfadens whose vertices are all the paragraphs, or one can do it directly as it is done in (Bott and Tu 1982):



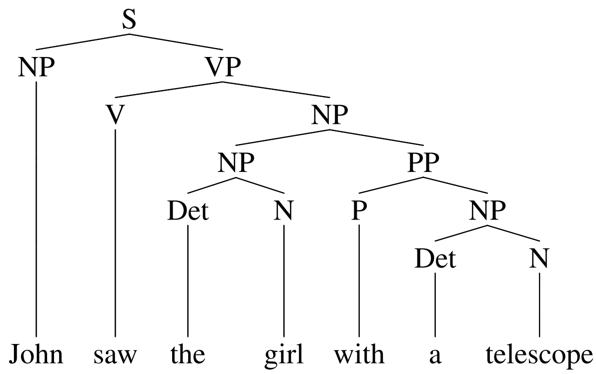
Indeed, the authors presuppose here the linear reading of paragraphs 1-6, 8-11, 13-16 and 20-22, which may be drawn explicitly. Likewise, one may go further by doing the next step. Given an admissible text, one can, by means of analytical reading or perhaps with the help of the author, find all the basis sets U_x of the phonocentric topology at the level of text and then draw the Hasse diagram of the corresponding poset. Certainly, the author has some clear representation of this kind during the writing process. Anyhow, the representations of this kind appear implicitly during the reading process at each semantic level.

These considerations may be repeated with slight modifications in order to define a *phonocentric* topology at the semantic level of sentence. Recall that we consider a sentence as a sequence of its words. So we have immediately the topology defined by applying the functor L to the linear order of $x \leq y$ of the ordinary sentence reading. In order to find the underlying phonocentric topology, we need to distinguish at the level of sentence its significant fragments or syntagmas being analogue to the meaningful parts at the level of text. In other words, we need to modify properly the criterion of meaningfulness. It is well known that yet a problem to define formally the notion of *word* is rather difficult. However, the goal of this paper is not to establish all the needed foundations from the very beginning but to reveal the topological structures underlying the natural language sentence understanding. Therefore, in speaking further about words and syntagmas, we accept for a moment a naive point of view of a descriptive grammar and we proceed by means of examples and explanations. Let us adopte as example the sentence "John saw the girl with a telescope" borrowed from (Werning 2003: 10). In applying the functor L to the linear word order of this sentence, we get a topology which doesn't catch the difference between two possible interpretations of this sentence: one interpretation where the seeing is by means of a telescope, and the second interpretation where the girl has a telescope. In the case of real spontaneous speech act, these two possible interpretations are easily distinguished by means of sentence intonation, stress, rhythm and melodic mouvement. When reading such a sentence, one needs some context to eliminate its ambiguity. Syntactically, these two interpretations correspond to the following different parse trees used in Chomsky's generative grammar:

$\psi_1 =$



$\psi_2 =$



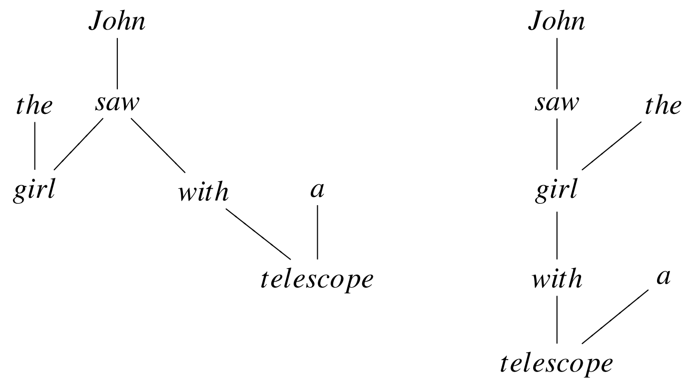
Recall that a *rewrite grammar* G is a quadruple $G = (T, C, S, R)$, where T and C are disjoint finite sets of symbols (called the *terminal* symbols and *non-terminal* symbols (clause-indicators) respectively), $S \in C$ is a distinguished non-terminal called the *start symbol*, and R is a finite set of *rewriting rules* or *productions*. A production is a paire (α, β) where α is a nonempty string of symbols of C and β is any string of symbols of $C \cup T$; productions are usually written $\alpha \to \beta$.

To get a sentence w of the language we start from S and apply the productions until no non-terminals are left. The sequence of productions used constitutes a *derivation* of w and we say that the grammar G generates w. In this way, we obtain the language L_G generated by the given grammar G; it consists of the strings on T that can be reached by a derivation from S (*grammatical strings*), and no others (i.e., *ungrammatical strings*). A grammar G is said to be a *context-free* iff all productions have the form $\alpha \to \beta$, where α is a string of a single non-terminal symbol, β is a non-empty string of symbols of $C \cup T$; this means that α is replaced by β independently of the context, i.e. independently of symbols on the both sides of α.

We say that a context-free grammar $G = (T, C, S, R)$ generates the labeled tree ψ iff the root node of ψ is labeled S, and for each node n in ψ, either n has no children and it is labeled with a terminal symbol or else there is a production $\alpha \to \beta$ in R where the label of n is α and the left-to-right sequence of labels of n's immediate children is β. A context-free grammar G generates a string w of terminals iff G generates a tree ψ whose *yield* (i.e., the left-to-right sequence of terminal symbols labeling ψ's leaf nodes) is w; ψ is called a *parse tree* of w (with respect to G). We define Y to be a function which maps trees to their yields. It is easy to see that the above two trees ψ_1 and ψ_2 with the yields $\mathsf{Y}(\psi_1) = \mathsf{Y}(\psi_2) =$ *John saw the girl with a telescope* are both generated by the grammar $G = (T, C, S, R)$, where $T = \{John, saw, the, girl, with, a, telescope\}$, $C = \{\mathrm{S}, \mathrm{NP}, \mathrm{N}, \mathrm{Det}, \mathrm{VP}, \mathrm{V}, \mathrm{PP}, \mathrm{P}\}$ and $R = \{\mathrm{S} \to \mathrm{NP\ VP}, \mathrm{NP} \to John, \mathrm{NP} \to \mathrm{Det\ N}, \mathrm{NP} \to \mathrm{NP\ PP}, \mathrm{N} \to girl, \mathrm{N} \to telescope, \mathrm{Det} \to a, \mathrm{Det} \to the, \mathrm{VP} \to \mathrm{VP\ PP}, \mathrm{VP} \to \mathrm{V\ NP}, \mathrm{V} \to saw, \mathrm{PP} \to \mathrm{P\ NP}, \mathrm{P} \to with\}$. Informally, N rewrites to nouns, NP rewrites to noun phrases, Det to determiners, V to verbs, VP rewrites to verb phrases, P to prepositions, PP to prepositional phrases.

The yield function Y gives rise to a string w of words with an ordinary linear order. The Hasse diagram of this linearly ordered set is a simple vertical chain. The trees ψ_1, ψ_2 are mapped to the same sentence w, though they are different and their difference reflects in some way the difference between two constructions of the sentence. Thus the question is to improve the yield function in order to make it sensible to tree's construction. Properly defined, this function, say W, should return the specialization order of the underlying phonocentric topology; that is, $\mathsf{W}(\psi) = Q(X)$ where X is the corresponding space with a phonocentric topology. It should be clearly $\mathsf{W}(\psi_1) \neq \mathsf{W}(\psi_2)$ for the aforesaid example. The Hasse diagrams of the two resulted specialization orders reflect the difference between the two parsing trees. It seems that the interpretation process should include a such reconstruction procedure besides the other operations.

There is a simple algorithm to transforme any such a parse tree into a Hasse diagram of some order structure on the underlying string of words (sentence), so that the corresponding topology is stronger than the topology of the linear order. First, we define how to transforme each type of elementary subtrees corresponding to rewriting rules of R. Secondly, we define how these elementary posets are grafted one after the other to give the resulted poset. Adopting some obvious definitions of these two steps, we define a function W which transformes the different parse trees ψ_1, ψ_2 into the different posets $\mathsf{W}(\psi_1)$, $\mathsf{W}(\psi_2)$ with the following Hasse diagrams:



The parse tree structures of Chomsky's generative grammar are useful in drawing the Hasse diagrams in the case of "fixed" word order languages such as English. Perhaps, the *stemma* formalism of Tesnière (1959) is more appropriate to be modified in order to capture the syntax of "free" word order languages such as Russian. This formalism, however, needs to be improved, if we want to conserve the continuity of the identical map $L(X,\preceq) \to L(X,\leq)$ considered above. On the other hand, yet for the "strict" word order languages, it would be more adequate to redefine the notion of primitive elements, i.e., consider *morphemes* instead of words. In this case, some rewriting rules become *context-sensitives*. It implies that we have to study context-sensitive languages. There are also another type of arguments which show that a natural language such as English is not context-free (Cohn 1980: 352).

Thus, at the level of sentence, the problem of the explicit phonocentric topology definition cannot be solved wholly within the Chomsky's generative grammar theory. However, the generative grammar is a formal device to provide a relatively extensive stock of well-formed expressions of a natural language. Its nomination as context-free or context-sensitive relates to the set R of rewriting rules, the particular set of admissible steps in the process of construction of a well-formed (grammatical) sentences, which should not be confused with the contextuality (and compositionality) principles involved in the interpretative process. In the sheaf-theoretic semantics, a rigorous formulation of these principles is based on the natural phonocentric topology defined at each semantic level. Following a generative grammar theory, a sentence is really represented by its parse tree ψ which yields a string $\mathsf{Y}(\psi)$ of its words with an ordinary linear order. We claim that there exists another function W defined on trees, which returns a specialization order corresponding to the phonocentric topology on the underlying word sequence of a sentence.

A reader's competence presumes a good skill both in the construction and in the interpretation of a sentence. Surely, the reading process involves a step by step decision about the syntactic well-formedness of a sentence. In other words, when reading a string of words "Colorless green ideas sleep furiously", a competent reader will easily recognizes that it does have a well-formed grammatical structure, contrary to another string, e.g., "*Furiously sleep ideas green colorless". The former isolated string of words was proposed by N. Chomsky in 1957 as an example of presumed meaningless grammatical string. This attribution was contested later by R. Jakobson who pointed out how it might be interpreted. We have restricted our sheaf-theoretic formal semantics exclusively to the class of admissible textes, i.e. textes written with good grace and intended for human understanding. This kind of the author's sincerity is usually supposed by the reader during the interpretative process. Nevertheless, at the semantic level of sentence, any input string of words will be interpreted in reading word by word, until the final punctuation point. This sequential process should only be blocked when the reader discovers that the input string fails to be grammatical. On the contrary, any grammatical string would imperatively receive an interpretation, but this interpretation would be dependent of the context. This phenomenon was noticed and played up by many linguistes, philosophers and writers who proposed sentences like Chomsky's famous example. So L. Tesnière considered a sentence "Le silence vertébral indispose la voile licide", B. Russell analyzed "Quadruplicity drinks procrastination", the Estonian writer E. Vilde wrote "Rabbits were running to the Eternity holding rainpipes in their mouths".

We have defined above the phonocentric topology at the level of text. Now, we would like to sketch how and why the definition of a phonocentric topology at the level of sentence may be recovered from the sentence parsing tree.

Firstly, the main symbolic containers N and V of a parse tree have an informal semantic definition, i.e. in terms of meaning accepted as a communicative content. Namely, N rewrites to noun usually defined as a person, thing or place; V rewrites to verb, defined as action or state. The other symbolic containers are defined in terms of subordination and coordination relations and their descriptions involve also an account of meaning. It explains why any parse tree of a grammatical string contains certain lexical and positional data involved in the interpretative process.

Now, for a given parse tree ψ, we explain how some parts of the yield string w should be defined as being analogue to the meaningful parts of a text. Notice first that the yield function Y may be clearly generalized to a function Y' which projets an arbitrary set of nodes of a parse tree ψ into the string w; this function Y' projets clearly the set Ψ of all nodes of ψ onto the string w, that is $\mathsf{Y}'(\Psi) = \mathsf{Y}(\psi) = w$. Now given a concrete parse tree ψ, for each node n labeled with a terminal, we can easily determine the set U_n of all the nodes which are needed for the interpretation of n, as we have sketched in the example above; this operation concerns some nodes of the parse tree and the used rewriting rules. This set U_n is projected by Y' to some substring of w. The set of all substrings of the kind $\mathsf{Y}'(U_n)$ will define a basis of the phonocentric topology on the yield string w of terminal symbols, and hence the corresponding specialization order denoted above as $\mathsf{W}(\psi)$; the latter is represented by its Hasse diagram.

At the semantic level of word, the phonocentric topology on a word (considered as a sequence of its syllables) seems to coincide with the topology assigned by applying the functor L to the linear order of syllables. As J. Dubois writes:

> *Ainsi lorsque l'on a énoncé la syllabe* ca- *un nombre important de mots sont encore possibles, et cependant de très nombreux termes sont déjà exclus. Si l'on énonce une deuxième syllabe* capi-*, on obtient pour la syllabe suivante une probabilité plus grande puisque, dans l'ensemble significatif des mots commençant par ce deux syllabes, je trouve* capitale, capital, capiton, *etc. Si j'énonce* capital *– je trouve alors un nombre plus restreint de mots (*capitalisme, capitaliste, *etc.) et la probabilité de la syllabe suivante est encore plus grande.* (Dubois 1969: 10)

He continues then by some statistical reasoning which is less important to our purposes; however, this example carries conviction that, at the level of word, the specialization order coincides with the ordinary linear order of reading. It is obvious in the situation of speech understanding in a conversation, where a listener hears a sequence of acoustic signals which should be build into words and then sentences. In the situation of text understanding, where a reader can catch at once the visual identity of many short words, the word recognition seems to be a more versatile activity because there are also many polysyllable words which cannot be caught at a glance in the reading process, especially when hyphenated.

As a final remark, we note that the systematic interpretation of linguistic notions in terms of topology and order and their further geometric study may be thought of as a kind of *formal textual syntax* because the word syntax originates from the Greek words $\sigma\upsilon\nu$ (together) and $\tau\alpha\xi\iota\zeta$ (sequence/order).

4. Sheaves of fragmentary meanings

Let X be an admissible text, and let $\mathscr{F}$ be an adopted *sense* or *mode of reading*. For a given fragment $U \subset X$, we collect all the fragmentary meanings of U in the set $\mathscr{F}(U)$. Thus we are given a map $U \mapsto \mathscr{F}(U)$ defined on the set $\mathscr{O}(X)$ of all opens $U \subset X$. Formulated not only for the whole text X but more generally for any meaningful part $V \subset X$, the precept of the hermeneutic circle 'to understand any part of text in accordance with the understanding of the whole text' defines a family of maps $\mathrm{res}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$, where $U \subset V$, such that $\mathrm{res}_{V,V} = \mathrm{id}_{\mathscr{F}(V)}$ and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ for all nested opens $U \subset V \subset W$. From a mathematical point of view, the data $(\mathscr{F}(V), \mathrm{res}_{V,U})_{V,U \in \mathscr{O}(X)}$ is a *presheaf* of sets (of fragmentary meanings) over X.

The reading process of a given fragment U is modeled as its (open) covering by some family of subfragments $(U_j)_{j \in J}$, where each U_j is supposed to be read in a distinct physical act.

According to Quine, there is no entity without identity. The definition of equality that seems to be quite adequate to our linguistic intuition is posed by the following:

Claim S (Separability). *Let X be an admissible text, and let U be a fragment of X. Suppose that $s, t \in \mathscr{F}(U)$ are two fragmentary meanings of U and there is an open covering $U = \bigcup_{j \in J} U_j$ such that $res_{U,U_j}(s) = res_{U,U_j}(t)$ for all U_j. Then $s = t$.*

Following a standard sheaf-theoretical terminology, a presheaf satisfying the claim (S) is called *separated*. Thus any sense (mode of reading) defines some *separated presheaf* $\mathscr{F}$ of fragmentary meanings over an admissible text X.

Following the precept of the hermeneutic circle 'to understand the whole text by means of understandings of its parts' this separated presheaf $\mathscr{F}$ should satisfy the following:

Claim C (Compositionality). *Let X be an admissible text, and let U be a fragment of X. Suppose that $U = \bigcup_{j \in J} U_j$ is an open covering of U; suppose we are given a family $(s_j)_{j \in J}$ of fragmentary meanings, $s_j \in \mathscr{F}(U_j)$ for all fragments U_j, such that $res_{U_i, U_i \cap U_j}(s_i) = res_{U_j, U_i \cap U_j}(s_j)$. Then there exists some meaning s of the whole fragment U such that $res_{U,U_j}(s) = s_j$ for all U_j.*

In sheaf theory, a separated presheaf satisfying the claim (C) is called a *sheaf*. The precept of the hermeneutic circle 'to understand the whole text by means of understandings of its parts' is a kind of compositionality principle at the level of text that generalizes the Frege's classic one; so the fragmentary meanings should satisfy the following:

Definition (Frege's generalized compositionality principle). *A separated presheaf of fragmentary meanings naturally attached to any sense (mode of reading) of an admissible text is really a sheaf; its sections over any fragment of the text are the fragmentary meanings; its global sections are the meanings of the whole text.*

Recall that the elements of $\mathscr{F}(U)$ are usually called *sections over U* and the elements of $\mathscr{F}(X)$ are usually called *global sections*.

We note that the claim (S) guarantees the meaning s, whose existence is claimed by (C), to be unique as such. It is not so hard to see that these two conditions (S) and (C) needed for presheaf to be a sheaf are analogous to those two conditions needed for a binary relation to be functional. So the true formulation of Frege's compositionality principle does not demand functionality but its sheaf-theoretic generalization which states that any presheaf of fragmentary meanings naturally attached to an admissible text ought *de facto* to be a sheaf. The sheaves arise whenever some consistent local data glues in a global one.

Let us consider now any two senses (modes of reading) $\mathscr{F}$, $\mathscr{G}$ of a given text X. The reader should become at home with the senses treated as functors although we call them sometimes as 'modes of readings' instead of 'senses' not only to stress the character of intentionality of each actual process of reading but rather to avoid a possible confusion which may be caused by another technical acceptation of the term 'sense'. So one can think, for example, about the historical sense $\mathscr{F}$ and the moral sense $\mathscr{G}$ of some biographical text. Let $U \subset V$ be any two fragments of the text X. It seems to be very natural to consider that any meaning s of fragment V understood in the historical sense $\mathscr{F}$ gives a certain well-defined meaning $\phi(V)(s)$ of the same fragment V understood in the moral sense $\mathscr{G}$. Hence, for each $V \subset X$, we are given a map $\phi(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$. To transfer from the meaning s of V in the historical sense to its meaning $\phi(V)(s)$ in the moral sense and then to restrict the latter to a subfragment $U \subset V$ is the same operation as to make first the restriction from V to U of the meaning s in the historical sense, and to make then a change of the historical sense to the moral one. This can be expressed in a simple way by saying that the following diagram

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathscr{G}(V) \arrow[d, "\mathrm{res}'_{V,U}"] \\ \mathscr{F}(U) \arrow[r, "\phi(U)"'] & \mathscr{G}(U) \end{tikzcd}$$

commutes for all fragments $U \subset V$ of X. We meet this situation of somebody's interpretation transfer from one mode of reading to another, or from understanding in one sense to understanding in some another sense many times a day. This kind of transfer from the understanding in one sense $\mathscr{F}$ to the understanding in another sense $\mathscr{G}$ is a usual matter of linguistic communication.

This notion of morphism is very near to that of *transformation incorporelle* of G. Deleuze and F. Guattari illustrated by several examples, one of which we quote:

> *Dans un détournement d'avion, la menace du pirate qui brandit un revolver est évidemment une action ; de même l'exécution des otages si elle a lieu. Mais la transformation des passagers en otages, et du corps-avion en corps-prison, est une transformation incorporelle instantanée, un* mass-media act *au sens où les Anglais parlent de* speech-act. (Deleuze and Guattari: 102–103)

To adapt this example, we need only to transform it into some written story about a hijacking. Hence, the family of maps $(\phi(V))_{V\in\mathscr{O}(X)}$ defines a change of mode of reading of a given text X, or simply a morphism $\phi: \mathscr{F} \mapsto \mathscr{G}$. It is obvious that a family of identical maps $\mathrm{id}_{\mathscr{F}(V)}: \mathscr{F}(V) \to \mathscr{F}(V)$ given for each open $V \subset X$ defines the identical morphism of the sheaf $\mathscr{F}$ which will be denoted as $\mathrm{id}_{\mathscr{F}}$. The composition of morphisms is defined in an obvious manner: for any two morphism $\phi: \mathscr{F} \mapsto \mathscr{G}$, $\psi: \mathscr{G} \mapsto \mathscr{H}$, we define $(\psi \circ \phi)(V) = \psi(V) \circ \phi(V)$. It is clear that this composition is associative every time it may be defined.

Thus, given an admissible text X, the data of all sheaves $\mathscr{F}$ of fragmentary meanings together with all its morphisms constitutes some *category* in a strict mathematical sense of the term. We name this category of particular sheaves describing the exegesis of a text X as a *category of Schleiermacher* and denote it as $\mathbf{Schl}(X)$ because he is often considered as the author of the *hermeneutic circle* principle of interpretation (Skirbekk & Gilje 1999). The part is understood in terms of the whole and the whole in terms of the parts. This part-whole structure in the understanding, he claimed, is principal in matter of interpretation of any text in natural language. The theoretical principle of *hermeneutic circle* is a precursor to these of *compositionality* and *contextuality* formulated later in 19th century. The succeeded development of hermeneutics has confirmed the importance of Schleiermacher's concept of circularity in text understanding. From our viewpoint, the concept of part-whole structure expressed by Schleiermacher as the hermeneutic circle principle, in the linguistic form, reveals the fundamental mathematical concept of a *sheaf* formulated by Leray more than a hundred years later, in 1945. This justifies us to name the particular category of sheaves $\mathbf{Schl}(X)$ after Schleiermacher.

Note in conclusion that a thorough choice of basic definitions should clarify the whole theory. Perhaps the presence of different acceptances of *Sinn* in Frege's writings explains his hesitations about the compositionality principle described in the interesting article of Janssen (2001) on the history of Frege's contextuality and compositionality principles. If we use the term *Sinn* to formalize a notion of the *mode of presentation of reference*, it seems to be very doubtful that, for example, two such *modes of presentation* for the subexpressions were composable in some third *mode of presentation of reference* for the whole expression. On the contrary, it seems to be very natural that if we pose under the subexpressions what we have understood after have read them (i.e., their meanings), we can understand from this data what means the whole expression. So one needs to precise the terminological convention on *sense* and *meaning* in order to discuss rigorously compositionality and contextuality principles.

5. Contextuality

So far, we have defined only a notion of fragmentary meanings, i.e., we consider to be meaningful only parts of a given text which are open relative to the phonocentric topology. But the question arises whether the other parts of text (i. e. non-open ones) would be meaningful. To consider at each semantic level not only the meanings of open parts (fragments) but also the meanings of points of a corresponding topological space, we define a notion of *contextual meaning*. The classic precept "nach der Bedeutung der Wörter muss im Satzzusammenhange, nicht in ihrer Vereinzelung gefragt werden" of *Die Grundlagen der Arithmetik* (Frege 1884: X) is referred nowadays as Frege's *contextuality principle*. Frege had written it eight years before his famous work *Über Sinn und Bedeutung* of 1892, where he had introduced in semantics a terminological distinction between *Sinn* and *Bedeutung*, which became essential for his further research. In connection with this distinction of Frege's work of 1892, Church and Russell translate the term *Bedeutung* as *denotation*. To indicate Frege's *Sinn/Bedeutung* distinction, one translates in English *Bedeutung* as *denotation* or sometimes as *reference* if it stands for the named object and as *denoting* if it stands for the relation of naming. It's clear that a word in isolation (e.g., when in a dictionary) doesn't refer to a particular object (with precision up to several exceptions). But the Frege's *contextuality principle* claims more, as expressed in the English translation of T. M. V. Janssen (2001: 115) : "Never ask for the meaning of a word in isolation, but only in the context of a sentence". Likewise for the English translation of J. L. Austin: "never ask for the meaning of a word in isolation, but only in the context of a proposition" (Frege 1884: X^{e}). In the French translation of C. Imbert (Frege 1969: 122), this principle is expressed as: "On doit rechercher ce que les mots veulent dire non pas isolément, mais pris dans leur contexte". We note that 'Satzzusammenhange' means literally 'in relation to phrase'.

The contextuality principle is usually quoted in the formulation Frege gives in the Introduction to *Die Grundlagen der Arithmetik*. In the foregoing considerations of § 60, Frege gives more explanations upon:

> *Only in a proposition have the words really a meaning* (Bedeutung)*. It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a sense* (Sinn)*; it is this that confers on its parts also their content.*[4] (Frege 1884: 71^{e})

It seems however that the *Sinn/Bedeutung* distinction from *Über Sinn und Bedeutung* of 1892 is not relevant here. This passage justifies rather the translation of *Bedeutung* with *meaning* and exemplifies also one of the Fregean acceptances of *Sinn* which corresponds to our *fragmentary meaning*: for some meaningful expression, "it is this that confers on its parts also their content". So we consider this Fregean formulation as an implicit definition which we have to precise if we want to recover the contextuality as a rigorous notion. On the other hand, we will complete our notion of *fragmentary meaning* in order to consider (at each semantic level) not only the meanings of fragments (or subexpressions) but also the meanings of its primitive (elementary) parts.

To elaborate our definition, we start by literal rephrasing the classic Frege's contextuality principle at the level of text as the claim that a given sentence has a meaning in relation to the whole text. But at the level of text this maximal contextualization seems to be excessive because our everyday practice of text and discourse understanding reveals that our understanding progress usually with the reading or listening, and the meanings of its sentences are caught during this process. In other words, the understanding of any sentence is not postponed until the reading of the final word of the whole text. To understand a given sentence x, we need a context constituted by some open (i.e., meaningful) neighborhood of x.

---

[4] English translation by J. L. Austin.

Let $\mathscr{V}(x)$ be the set of all open neighborhoods of a sentence x. Let $U,V \in \mathscr{V}(x)$ and let $\mathscr{F}$ be be an adopted sense. Two fragmentary meanings $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$ seem to induce the same contextual meaning to the sentence x if s and t agree on some smaller meaningful fragment which contains x. It seems to be conform with the common reader's intuition about what would it means then two given fragmentary meanings s and t define the same contextual meaning for x. So these two fragmentary meanings $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$ are said to induce the same contextual meaning at x if there exists some open neighborhood W of x, such that $W \subset U \cap V$ and $\mathrm{res}_{U,W}(s) = \mathrm{res}_{V,W}(t) \in \mathscr{F}(W)$.

This property should be demanded by any reasonable definition for the notion of contextual meaning. This relation 'induce the same contextual meaning at x' is obviously an equivalence relation, and any equivalence class of fragmentary meanings agreeing in some open neighborhood of x is called a *contextual meaning* of x.

According to standard sheaf-theoretic terminology, the set of all equivalence classes is called a *stalk* of $\mathscr{F}$ at x and is denoted by $\mathscr{F}_x$. The equivalence class of a fragmentary meaning $s \in \mathscr{F}(U)$ in $\mathscr{F}_x$ is called the *germ of s at x* and is denoted by $\mathrm{germ}_x s$. This definition for the set $\mathscr{F}_x$ of all contextual meanings of the sentence x is a variant of what is well known as a construction for *inductive limit* explained in many standard sources on the category theory. Recalling this construction, we postulate at the level of text the following

Definition (Frege's generalized contextuality principle). *Let X be an admissible text and x be a sentence within a fragment $U \subset X$; a contextual meaning of x is defined as the germ at x of some fragmentary meaning $s \in \mathscr{F}(U)$, where the sheaf $\mathscr{F}$ is the adopted sense (mode of reading); the set $\mathscr{F}_x$ of all contextual meanings of a sentence $x \in X$ is defined as the stalk of $\mathscr{F}$ at x, i.e., as the inductive limit* $\mathscr{F}_x = \varinjlim(\mathscr{F}(U), \mathrm{res}_{U,V})_{U,V \in \mathscr{V}(x)}$.

Intuitively, the notion of contextual meaning is almost obvious. The problem is that the same sentence may occur in many quite different texts; likewise, the same word may occur in many quite different sentences. Following Frege, in seeking the meaning of a word, we must consider it in the context of some sentence; likewise, in seeking the meaning of a sentence, we must consider it in the context of some text. Suppose we want to assign a contextual meaning to some sentence x of a text X. Given a fragmentary meaning s of a fragment U containing x, we dispose a piece of data which creates some context to determine a *contextual meaning* of x as $\mathrm{germ}_x s$.

According to a well-known *inductive limit characterizing theorem* (Tennison 1975: 5), this contextuality principle stated at the level of text is equivalent to the conjunction (Ct)&(E) of the following two claims:

Claim Ct (Contextuality). *Let $\mathscr{F}$ be a mode of reading (sense) adopted for a given text X, then for any contextual meaning $f \in \mathscr{F}_x$ of a sentence x, there exist a neighborhood U of x and a fragmentary meaning $s \in \mathscr{F}(U)$ such that $f = \mathrm{germ}_x s$.*

The claim (Ct) is a generalization in the narrow sense of the Frege's classic contextuality principle; it may be paraphrased as "ask for the meaning of a sentence only in the context of some fragment of a given text".

Claim E (Equality). *Let U, V be two open neighborhoods of a sentence x and let $s \in \mathscr{F}(U)$, $t \in \mathscr{F}(V)$ be two fragmentary meanings for a given mode of reading (sense) $\mathscr{F}$. Then the equality $germ_x s = germ_x t$ in $\mathscr{F}_x$ between induced contextual meanings of the sentence x holds if and only if there exists an open neighborhood W of x such that $W \subset U$, $W \subset V$ and $res_{U,W}(s) = res_{V,W}(t)$.*

The claim (E) is an *explicit criterion of equality* between contextual meanings of a given sentence in the context of a given text.

The generalized contextuality principle proposed above is an explicit definition of contextual meanings at the semantic level of text. A similar definition may be given for contextual meanings at each semantic level. This one formulated at the level of sentence renders Frege's classic contextuality principle. As soon as the semantic level is fixed, the corresponding contextual meaning of $x \in X$ is defined as $\mathrm{germ}_x s$, where $s \in \mathscr{F}(U)$ of some open neighborhood U of x.

Remark 1. More generally, one can consider the inductive system of all the open neighborhoods of some arbitrary part $A \subset X$ to define all its contextual meanings as $\mathscr{F}_A = \varinjlim_{U \supset A} (\mathscr{F}(U))$; as it was to be expected, for any open part A, this definition provides its contextual meanings as the fragmentary ones already given, i.e., in this case $\mathscr{F}_A = \mathscr{F}(A)$; in particular, for any minimal element x being open singleton, we have $\mathscr{F}_x = \mathscr{F}(\{x\})$.

Remark 2. The inductive limit may be calculated by means of any confinal part of a considered inductive system; in particular, $\varinjlim(\mathscr{F}(U), \mathrm{res}_{U,V})_{U,V \in \mathscr{V}(x)}$ is clearly determined by the basis of neighborhoods in $\mathscr{V}(x)$, that is, by the smallest basic set U_x. One sees clearly that the stalk $\mathscr{F}_x$ is in a one-to-one correspondence with $\mathscr{F}(U_x)$. Although the set $\mathscr{F}_x$ of contextual meanings of x can be identified with the set of fragmentary meanings of the smallest open neighborhood U_x of x, we need the conceptual definition of $\mathscr{F}_x$ as $\varinjlim(\mathscr{F}(U), \mathrm{res}_{U,V})_{U,V \in \mathscr{V}(x)}$ to prove the theorems. Moreover, the understanding is progressing in time during the consecutive process of reading and rereading, where the different fragmentary meanings of the different neighborhoods of x are identified to give finally one contextual meaning of x, which is a $\mathrm{germ}_x(s)$ of some $s \in \mathscr{F}(U)$ grasped by the reader in some particular situation of reading. This neighborhood U of x need not be the smallest one U_x because it is not explicitly marked for each x as being the minimal context needed for its understanding.

Now, to illustrate the notion of contextual meaning with an example, let us consider a simple sentence x like "John is a philosopher". In some conversation, it might mean a lot of things, according to the context of its usage. One can say it and, at the same time, think something more or something else as for example: "John is a philosopher, for he has published in *Studia Philosophica* a long article on phenomenology", or "John is a philosopher, for he lets reason govern his life", or simply "John is a philosopher, for he studies philosophy at Harvard University". Anybody who participates in these communicative situations, surely in the context understands what the simple "John is a philosopher" means in each particular utterance. Let us now consider a possible contextual meaning of this phrase written in some text. In this case, however, the author ought to write some meaningful fragment U containing x = "John is a philosopher" (some open neighborhood of x) in order to make clear and understandable what x means. The smallest such a neighborhood U_x depends on the particular author's communicative intention and, in general, this U_x cannot be reduced to x. Hence, the grasped contextual meaning of x corresponds to one of the fragmentary meanings of U_x which is grasped in accordance with the reader's mode of reading (sense) $\mathscr{F}$, and it cannot be otherwise! Paraphrasing Frege, we say: "Never ask for the meaning of a sentence in isolation, but only in the context of some fragment of a text". It may be U_x or some other neighborhood of x, but *de facto* the contextual meaning of x can be identified with the $\mathrm{germ}_x(s)$ of some fragmentary meaning $s \in \mathscr{F}(U)$. For any other sentence $y \in U_x$ such that $y \neq x$, we have $U_y \neq U_x$, and hence the contextual meaning of y is defined by one of the fragmentary meanings of U_y, not of U_x, despite of the fact that y lies in U_x. So the process of the reader's understanding may be thought of as the consecutive choice of only one element from each stalk $\mathscr{F}_x$. Even for a juridical text, the multiplicity of possible contextual meanings in $\mathscr{F}_x$ is inevitable.

For the coproduct $F = \bigsqcup_{x \in X} \mathscr{F}_x$, we define now a map $p\colon F \to X$ as $p(\mathrm{germ}_x s) = x$ which we call a *projection*. Every fragmentary meaning $s \in \mathscr{F}(U)$ determines a function $\dot{s}\colon x \mapsto \mathrm{germ}_x s$ to be well-defined on U; for each $x \in U$, its value $\dot{s}(x)$ is taken in $\mathscr{F}_x$. This gives rise to functional representation $\eta(U)\colon s \mapsto \dot{s}$ for all fragmentary meanings $s \in \mathscr{F}(U)$.

We define the topology on F by taking as a basis of open sets all the image sets $\dot{s}(U) \subset F$. Given a fragment $U \subset X$, a continuous function $t\colon U \to F$ such that $t(x) \in p^{-1}(x)$ for all $x \in U$ is called a *cross-section*. For any cross-section $t\colon U \to F$, the projection p has the obvious property $p(t(x)) = x$ for all $x \in U$. The topology defined on F makes p and every cross-section of the kind $\dot{s}$ continuous. So we have defined two topological spaces F, X and a continuous map $p\colon F \to X$. In topology, this data (F, p) is called a *bundle over the base space* X. A *morphism* of bundles from $p\colon F \to X$ to $q\colon G \to X$ is a continuous map $h\colon F \to G$ such that the diagram

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes, that is $q \circ h = p$.

We have so defined a category of bundles over X. A bundle (F, p) over X is called *étale* if $p\colon F \to X$ is a local homeomorphism. All étale bundles and their morphisms constitute a full subcategory in the category of bundles over X. It is immediately seen that a bundle of contextual meanings $(\bigsqcup_{x \in X} \mathscr{F}_x, p)$ constructed as above from a given sheaf $\mathscr{F}$ of fragmentary meanings is étale. Thus, for an admissible text X, we have defined the category $\mathbf{Context}(X)$ of étale bundles of contextual meanings over X as a framework for the generalized contextuality principle at the level of text.

The similar definition may be formulated at each semantic level. This one formulated at the level of sentence renders Frege's classic contextuality principle. As soon as the semantic level is fixed, the definition of a contextual meaning for a point x a corresponding topological space X is given as $\mathrm{germ}_x s$, where s is some fragmentary meaning defined on some neighborhood U of x.

6. Frege duality

For a given admissible text X, we have defined two categories formalizing the interpretative process: the Schleiermacher category $\mathbf{Schl}(X)$ of *sheaves* of fragmentary meanings and the category $\mathbf{Context}(X)$ of *étale bundles* of contextual meanings. Our intention now is to relate them to each other.

We will firstly define a so-called *germ-functor* $\Lambda\colon \mathbf{Schl}(X) \to \mathbf{Context}(X)$. For each sheaf $\mathscr{F}$, it assigns an étale bundle $\Lambda(\mathscr{F}) = (\bigsqcup_{x\in X} \mathscr{F}_x, p)$, where the projection p is defined as above. For a morphism of sheaves $\phi\colon \mathscr{F} \to \mathscr{F}'$, the induced map of stalks $\phi_x\colon \mathscr{F}_x \to \mathscr{F}'_x$ gives rise to a continuous map $\Lambda(\phi)\colon \bigsqcup_{x\in X} \mathscr{F}_x \to \bigsqcup_{x\in X} \mathscr{F}'_x$ such that $p' \circ \Lambda(\phi) = p$; hence $\Lambda(\phi)$ defines a morphism of bundles. Given another morphism of sheaves ψ, one sees easily that $\Lambda(\psi \circ \phi) = \Lambda(\psi) \circ \Lambda(\phi)$ and $\Lambda(\mathrm{id}_{\mathscr{F}}) = \mathrm{id}_F$. Thus, we have constructed a desired germ-functor $\Lambda\colon \mathbf{Schl}(X) \to \mathbf{Context}(X)$.

We will now define a so-called *section-functor* $\Gamma\colon \mathbf{Context}(X) \to \mathbf{Schl}(X)$. We denote a bundle (F, p) over X simply by F. For a bundle F, we denote the set of all its cross-sections over U by $\Gamma(U,F)$. If $U \subset V$ are open, one has a restriction map $\mathrm{res}_{V,U}\colon \Gamma(V,F) \to \Gamma(U,F)$ which operates as $s \mapsto s|_U$, where $s|_U(x) = s(x)$ for all $x \in U$. It's clear that $\mathrm{res}_{U,U} = \mathrm{id}_{\Gamma(U,F)}$ for any open U, and that the transitivity $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ holds for all nested opens $U \subset V \subset W$. So we have constructed obviously a sheaf $(\Gamma(V,F), \mathrm{res}_{V,U})$.

For any given morphism of bundles $h\colon E \to F$, we have a map $\Gamma(h)(U)\colon \Gamma(U,E) \to \Gamma(U,F)$ defined as $\Gamma(h)(U)\colon s \mapsto h \circ s$, which is obviously a morphism of sheaves. Thus, we have constructed a desired section-functor $\Gamma\colon \mathbf{Context}(X) \to \mathbf{Schl}(X)$.

The fundamental theorem of topology states that the section-functor Γ and the germ-functor Λ establish a *dual adjunction* between the category of presheaves and the category of bundles (over the same topological space); this dual adjunction restricts to a *dual equivalence* of categories (or *duality*) between corresponding full subcategories of sheaves and of étale bundles (Lambek and Scott 1986: 179, Mac Lane and Moerdijk 1992: 89). In the linguistic situation, this important result yields the following:

Theorem (Frege duality). *The generalized compositionality and contextuality principles are formulated in terms of categories that are in natural duality*

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$$

established by the section-functor Γ *and the germ-functor* Λ*, which are the pair of adjoint functors.*

Obtained by the same reasoning as many of well-known classic dualities such as Stone, Gelfand-Naimark, and Pontrjagin-van Kampen ones, Frege duality obviously may be formulated also at the semantic level of sentence and even of word, that gives rise to some *functional representation* of fragmentary meanings at each semantic level and permits to establish an *inductive theory of meaning* (Prosorov 2004, 2005b) describing how runs the process of text understanding. For more details, we refer the reader to our works (Prosorov 2004, 2005a, 2005b).

7. Inductive theory of meaning

In this section we give an outline of the inductive theory of meaning following our exposition of (2005b).

Let X be an admissible text of arbitrary length interpreted in some adopted mode of reading (sense) $\mathscr{F}$. The reading process consists in the open covering of X by some family $(U_j)_{j\in J}$ of fragments, each having been read during a single action. So one starts the $(i+1)$th resumption of the reading process by keeping in mind some fragmentary meaning $s \in \mathscr{F}(U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_i})$, where U_{j_1} is a fragment read firstly, and so on, and finally U_{j_i} is a fragment read lastly. This fragmentary meaning s were composed as an intermediate result of interpretation process according to sheaf-theoretical formulation of compositionality principle, and one starts to read the $(i+1)$th fragment $U_{j_{i+1}}$ in the context of having grasped s. So we need to describe the process of understanding of the fragment $U_{j_{i+1}}$. Recall that following our terminological convention, the open $U_{j_{i+1}}$ is a union of the minimal basic opens of the kind U_x. So the problem is reduced to explain how the reader grasps some fragmentary meaning of a minimal basic open of the kind U_x.

Usually one reads a given text in the ordinary linear order $\leq$ it bears on. It may occur to begin a reading from the passage already read. If this is the case, one arrives quickly to a coherent understandings for the part already read and continues the usual reading process. So we may suppose that $U_x \subset U_{j_{i+1}} \cap I(x)$, where $I(x) = \{l \mid l \leq x\}$.

Suppose that we have explained how the reader has grasped some fragmentary meaning s' of $U_x \cap I(x')$, where x' is the sentence immediately preceding x in the ordinary sentence order $\leq$. Following the functional representation of fragmentary meanings, this s' is represented by some sequence of contextual meanings of the sentences containing in $U_x \cap I(x')$. So we need to explain how the reader grasps some contextual meaning of x with the purpose to extend the sequence of grasped contextual meanings on the whole U_x. But during the process of reading of the sentence x at the level of sentence, where the corresponding Frege compositionality principle holds, the reader grasps some its global meaning at the level of sentence, which is apparently one of its literal meaning. This literal meaning of the sentence x together with the fragmentary meaning s' of $U_x \cap I(x')$ allows to grasp some fragmentary meaning of the whole U_x, and whence the contextual meaning of x. So the reader has extended the sequence of grasped contextual meanings to the whole U_x.

It was the *inductive step*. As for the *basis of induction*, note that the minimal elements of U_x (in the sense of the order $\preceq$) are open singletons. Following our *Remark 1*, for any such open singleton $\{y\}$, the set of all its contextual meanings at the level of text is in the bijective correspondence with the set of all its fragmentary meanings at the level of text, which is evidently the set of all global meanings of the sentence y at the level of sentence. This is a *recursive step* to the inferior semantic level, where the corresponding Frege compositionality principle holds to explain how the reader grasps one of its literary meaning. For $\{y\}$ being open, that is, meaningful at the level of text, its literal meaning grasped at the level of sentence is apparently its fragmentary (and equally contextual) meaning at the level of text. Note that the first sentence of a novel is always supposed to be understood in his own context, that is supposed to be open in the phonocentric topology.

8. Sheaf-theoretic semantics

Thus the true object of study in the natural language semantics should be a pair $(X,\mathscr{F})$, i.e. an admissible text X with a sheaf $\mathscr{F}$ of its fragmentary meanings; any such a couple is called a *textual space*. But this representation is possible only in the realm of a language following the famous slogan of Wittgenstein "to understand a text is to understand a language". Rigorously, this claim may be formulated in the frame of category theory. Likewise the present sheaf-theoretic formal semantics describes a natural language in the *category of textual spaces* **Logos**. The objects of this category are couples $(X,\mathscr{F})$, where X is a topological space naturally attached to an admissible text and $\mathscr{F}$ is a sheaf of fragmentary meanings defined on X; the morphisms are couples $(f,\theta)\colon (X,\mathscr{F}) \to (Y,\mathscr{G})$ made up of a continuous map $f\colon X \to Y$ and a f-morphism of sheaves θ which respects the concerned sheaves, i.e., $\theta\colon \mathscr{G} \to f_*\mathscr{F}$, where f_* is a well-known *direct image* functor. All these notions are discussed at length in our works of 2001-2005.

Given any admissible text X considered as fixed forever, it yields very naturally a full subcategory $\mathbf{Schl}(X)$ in the category **Logos** of all textual spaces. This category of Schleiermacher $\mathbf{Schl}(X)$ describes the situation when the reader is interested in the exegesis of some particular text as, for example, Sacred Scripture.

The inductive theory of meaning based on Frege duality, and also the different categories and functors related to discourse and text interpretation processes are the principal objects of study in the *sheaf-theoretic formal semantics* as we understand it.

Acknowledgements

I wish to express my gratitude to Urmas Sutrop, Institute of the Estonian Language, for inviting me to publish this article in the journal publishing works of the celebrated linguistic school of Tartu. This paper extends my contributed talk given at the Third International Workshop on Semantics, Pragmatics, and Rhetoric (SPR-05) Donostia - San Sebastián, November 10-12, 2005, in which my participation was partially supported by the Russian Foundation for Basic Research (grant no 05-06-85610).

Address:
Oleg Prosorov
St. Petersburg Department of V. A. Steklov Institute of Mathematics
27, Fontanka
191023 St. Petersburg
Russia
E-mail: prosorov@pdmi.ras.ru
Tel.: +7 (812) 3124058
Fax: +7 (812) 3105377

References

Bott, R. and L. W. Tu (1982) *Differencial forms in algebraic topology*. New York: Springer-Verlag.
Cohn, P. M. (1980) *Universal algebra*. In M. Hazewinkel, ed. Mathematics and its applications, vol. 6. Dordrecht: D. Reidel Publishing Company.

Deleuze, G. and F. Guattari (1980) *Capitalisme et schizophrénie 2. Mille plateaux*. (Coll. « Critique ».) Paris: Minuit.

Dubois, J. (1969) *Grammaire structurale du français : la phrase et les transformations.* (Coll. « Langue et Langage ».) Paris: Larousse.

Erné, M. (1991) "The ABC of order and topology". In H. Herrlich, H.-E. Porst, eds. *Research and exposition in mathematics, vol. 18. Category theory at work*, 57–83. Berlin: Heldermann.

Frege, G. (1884) *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*. Breslau: Verlag von W. Koebner. (Reprinted by Basil Blackwell, Oxford, 1953, English translation by J. L. Austin.)

Frege, G. (1960) "On Sense and Reference". In P. Geach, M. Black, eds. *Translations from the philosophical writings of Gottlob Frege*, 56–78. Oxford: Blackwell.

Frege, G. (1967) "*Begriffsschrift*, a formula language, modeled upon that of arithmetic, for pure thought" (S. Bauer-Mengelberg, trans.). In J. van Heijenoort, ed. *From Frege to Gödel*, 1–82. Cambridge, MA: Harvard University Press. (Original work published 1879.)

Frege, G. (1969) *Les fondements de l'arithmétique*. (C. Imbert, trans.) Paris: Seuil.

Janssen, T. M. V. (2001) "Frege, contextuality and compositionality". *Journal of Logic, Language, and Information* 10, 115–136.

Lambek, J. and P. S. Scott (1986) *Introduction to higher order categorical logic*. Cambridge: Cambridge University Press.

Mac Lane, S. and I. Moerdijk (1992) *Sheaves in geometry and logic. A first introduction to topos theory*. New York: Springer-Verlag.

Manin, Y. I. (1977) *A course in mathematical logic*. New York: Springer-Verlag.

May, J. P. (2003) *Finite topological spaces. Notes for REU.* Retrieved January 31, 2005, from http://www.math.uchicago.edu/ may/MISC/FiniteSpaces.pdf.

Prosorov, O. (2001) "Esquisse d'une herméneutique formelle". *Echos du Collège : Dialogue franco-russe* 2, 9–29.

Prosorov, O. (2002) *Herméneutique formelle et principe de Frege généralisé*. (PDMI preprint 4/2002.) St. Petersburg: Steklov Mathematical Institute.

Prosorov, O. (2003) *Formal hermeneutics and Frege duality*. (PDMI preprint 5/2003.) St. Petersburg: Steklov Mathematical Institute.

Prosorov, O. (2004) *Compositionnalité et contextualité, deux principes de Frege en adjonction*. (PDMI preprint 8/2004.) St. Petersburg: Steklov Mathematical Institute.

Prosorov, O. (2005a) "Compositionality and contextuality as adjoint principles". In M. Werning, E. Machery and G. Schurz, eds. *The compositionality of meaning and content, vol. 2. Applications to linguistics, psychology and neuroscience*, 149–174. Frankfurt: Ontos-Verlag.

Prosorov, O. (2005b) Formal hermeneutics based on Frege duality. In E. Maier, C. Bary and J. Huitink, eds. *Proceedings of SuB9*, 286–298. Nijmegen: NCS.

Rastier, F. (1995) "Communication ou transmission". *Césure* 8, 151–195.

Serre, J.-P. (1979) *Local fields*. (M. J. Greenberg, trans.) New York: Springer-Verlag.

Skirbekk, G. and N. Gilje (1999) *History of philosophy. An introduction to the philosophical roots of modernity.* Oslo: Scandinavian University Press.

Stanley, R. P. (1986) *Enumerative combinatorics, vol. 1.* Monterey, CA: Wadsworth & Brooks/Cole.

Sundholm, G. (2002) "A century of inference: 1837-1936". In P. Gärdenfors, J. Woleński and K. Kijania-Placek, eds. *In the scope of logic, methodology and philosophy of science. volume two of the 11th international congress of logic, methodology and philosophy of science, Cracow, August 1999*, 565–580. (Synthese Library, 316.) Dordrecht: Kluwer.

Tennison, B. R. (1975) *Sheaf theory*. Cambridge: Cambridge University Press.

Tesnière, L. (1976) *Éléments de syntaxe structurale*. (Deuxième éd. revue et corrigée.) Paris: Librairie C. Klincksieck. (Original work published 1959.)

Werning, M. (2003) *The reasons for semantic compositionality*. Retrieved January 24, 2006, from http://thphil.phil-fak.uni-duesseldorf.de/index.php/filemanager/download/212/ReasonsComp.pdf.