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On polynomials that are weakly uniformly continuous on the unit ball of a Banach space

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Abstract. We prove quantitative strengthenings of results on polynomials that are weakly uniformly continuous on the unit ball of a Banach space due to Aron, Lindström, Ruess, and Ryan (*Proc. Amer. Math. Soc.*, 1999, **127**, 1119–1125) and to Toma (*Aplicações holomorfas e polinômios τ-contínuos.* 1993). Our method is based on the uniform factorization of compact sets of compact operators.

Key words: Banach spaces, uniform compact factorization, n-homogeneous polynomials.

1. INTRODUCTION

Let X and Y be Banach spaces over the same, either real or complex, field $\mathbb{K}$. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y, and by $\mathcal{K}(X,Y)$ its subspace of compact operators.

Let $\mathcal{L}^s(^nX)$ denote the Banach space of continuous symmetric n-linear forms on X and let $\mathcal{P}(^nX)$ denote the Banach space of continuous n-homogeneous polynomials on X. Then for each $P \in \mathcal{P}(^nX)$ there is a unique $A_P \in \mathcal{L}^s(^nX)$ satisfying $P(x) = A_P(x,\dots,x)$ for each $x \in X$.

Recall that $P \in \mathcal{P}(^nX)$ is *weakly uniformly continuous* on the closed unit ball B_X of X if for each $\epsilon > 0$ there are $x_1^*,\dots,x_n^* \in X^*$ and $\delta > 0$ such that if $x, y \in B_X$, $|x_i^*(x-y)| < \delta$ for $i = 1,\dots,n$, then $|P(x) - P(y)| < \epsilon$. Let $\mathcal{P}_{wu}(^nX)$ denote the subspace of $\mathcal{P}(^nX)$ consisting of the polynomials that are weakly uniformly continuous on B_X. The corresponding subspace of $\mathcal{L}^s(^nX)$ is denoted by $\mathcal{L}^s_{wu}(^nX)$. Notice that $\mathcal{P}_{wu}(^nX)$, with the norm induced from $\mathcal{P}(^nX)$, is a Banach space (see [1], Proposition 2.4).

For each $P \in \mathcal{P}(^nX)$ there is a linear operator $T_P : X \to \mathcal{L}^s(^{n-1}X)$ defined by $(T_P x_1)(x_2, ..., x_n) = A_P(x_1, x_2, ..., x_n)$. Clearly, the correspondence $A_P \to T_P$ is linear and $\|T_P\| = \|A_P\|$. According to [1], $P \in \mathcal{P}_{wu}(^nX)$ if and only if $T_P \in \mathcal{K}(X, \mathcal{L}^s(^{n-1}X))$. Moreover, if $P \in \mathcal{P}_{wu}(^nX)$, then $T_P \in \mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$.

In 1999, Aron et al. (see [2], Proposition 5) proved the following result.

Theorem 1 [2]. *Let X be a Banach space and let $n = 2, 3, \ldots$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$. Then there exists a compact subset C of X^* such that for all $S \in C_n$ and all $x \in X$*

$$|(Sx)(x, ..., x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

Theorem 1 together with its proof in [2] gives no information about the size of the set C corresponding to the size of C_n.

The purpose of this article is to prove the following quantitative strengthening of Theorem 1. We denote $|C| = \sup\{\|x\| : x \in C\}$, where C is a bounded set in a Banach space.

Theorem 2. *Let X be a Banach space and let $n = 2, 3, \ldots$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$. Then there exists a compact circled subset C of X^* with $|C| = \max\{|C_n|, 1\}$ such that for all $S \in C_n$ and all $x \in X$*

$$|(Sx)(x, \ldots, x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

We use a standard notation. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The linear span of A is denoted by span A and the circled hull by $\mathrm{circ}A$.

2. PROOF OF THEOREM 2

The proof of Theorem 2 will be based on a factorization result that easily follows from

Lemma 1. *Let X and Y be Banach spaces. For every relatively compact subset C of $\mathcal{K}(X, Y)$, there exist a reflexive Banach space Z, a linear mapping $\Phi : \mathrm{span}\ C \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, Y)$ such that $S = v \circ \Phi(S)$ for all $S \in \mathrm{span}\ C$. The mapping Φ restricted to C is a homeomorphism and satisfies*

$$\|S\| \leq \|\Phi(S)\| \leq \min\{|C|,\ |C|^{1/2} b^{1/2} \|S\|^{1/2}\},$$

$S \in C$, *where $b \approx 2\frac{1}{2}$ is an absolute constant.*

Proof. Since $\overline{\operatorname{circ} C}$ is a compact subset of $\mathcal{K}\big(X, Y\big)$, by [3], Theorem 6, there exist a reflexive Banach space Z, a linear mapping $\Phi : \operatorname{span} C \to \mathcal{K}\big(X, Z\big)$, and a norm one operator $v \in \mathcal{K}\big(Z, Y\big)$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span} C$. Moreover, the mapping Φ restricted to $\operatorname{circ} C$ is a homeomorphism satisfying

$$\|S\| \leq \|\Phi(S)\| \leq \min\Big\{\, \frac{\mathrm{d}}{2},\ \Big(\frac{\mathrm{d}}{2}\Big)^{1/2} b^{1/2}\|S\|^{1/2}\Big\},$$

$S \in \operatorname{circ} C$, where $\mathrm{d} = \operatorname{diam} \operatorname{circ} C$.

Since for all $S \in C$

$$\|S\| = \frac{1}{2}\|2S\| = \frac{1}{2}\|S - (-S)\| \leq \frac{\mathrm{d}}{2},$$

we get $|C| \leq \mathrm{d}/2$. On the other hand, for all $S, T \in \operatorname{circ} C$, we have $S = \lambda S_0$ and $T = \mu T_0$ for some $S_0, T_0 \in C$ and for some $\lambda, \mu \in \mathbb{K}$ with $|\lambda|, |\mu| \leq 1$. Hence

$$\|S - T\| \leq \|S\| + \|T\| = \|\lambda S_0\| + \|\mu T_0\|$$

$$= |\lambda|\|S_0\| + |\mu|\|T_0\| \leq \|S_0\| + \|T_0\| \leq |C| + |C|,$$

$S, T \in C$. Therefore $\mathrm{d}/2 \leq |C|$. Consequently, $\mathrm{d}/2 = |C|$. □

The proof of Theorem 2 follows the idea of the proof of Proposition 5 in [2].

Proof of Theorem 2. We proceed by induction on $n = 2, 3, \ldots$. Let C_2 be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}(^1X)) = \mathcal{K}(X, X^*)$. By Lemma 1 there exist a Banach space Z, a linear mapping $\Phi : \operatorname{span} C_2 \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, X^*)$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} C_2$. Then for all $S \in C_2$ and all $x \in X$,

$$|(Sx)(x)| = |v(\Phi(S)x)(x)| = |(v^*x)(\Phi(S)x)|,$$

hence

$$|(Sx)(x)| \leq \|v^*x\|\|\Phi(S)x\|.$$

Put

$$C_\Phi = \overline{\{(\Phi(S))^*(z^*) : S \in C_2, z^* \in B_{Z^*}\}} \subset X^*.$$

Then C_Φ is circled. To prove that it is also compact, let us fix an arbitrary $\varepsilon > 0$. Let $\{\Phi(S_1), \ldots, \Phi(S_n)\}$, $S_k \in C_2$, be an ε-net in the relatively compact set $\{\Phi(S) : S \in C_2\}$. Since $\Phi(S_k)$ is a compact operator, $(\Phi(S_k))^*$ is also a compact operator and therefore $(\Phi(S_k))^*(B_{Z^*})$ is a relatively compact set. Since $\bigcup_{k=1}^n (\Phi(S_k))^*(B_{Z^*})$ is clearly a relatively compact ε-net in the set $\{(\Phi(S))^*(z^*) : S \in C_2, z^* \in B_{Z^*}\}$, this set is relatively compact. Hence, C_Φ is a compact set.

Moreover, we get

$$\|\Phi(S)x\| = \sup_{z^* \in B_{Z^*}} |z^*(\Phi(S)x)| = \sup_{z^* \in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \leq \sup_{x^* \in C_\Phi} |x^*(x)|$$

for all $S \in C_2$ and all $x \in X$.

Denoting

$$C_v = \overline{v(B_Z)} \subset X^*,$$

we have that C_v is circled and compact, and

$$\|v^* x\| = \sup_{z \in B_Z} |(v^* x)(z)| = \sup_{z \in B_Z} |(vz)(x)| \leq \sup_{x^* \in C_v} |x^*(x)|$$

for all $x \in X$.

Finally, let $C = C_\Phi \cup C_v$. Then C is circled and compact, and

$$|(Sx)(x)| \leq \|v^* x\| \|\Phi(S)x\| \leq \sup_{x^* \in C_v} |x^*(x)| \sup_{x^* \in C_\Phi} |x^*(x)| \leq \sup_{x^* \in C} |x^*(x)|^2$$

for all $S \in C_2$ and all $x \in X$.

By the definition of $|C|$,

$$\begin{aligned} |C| &= \sup_{x^* \in C} \|x^*\| = \sup_{x^* \in C_\Phi \cup C_v} \|x^*\| = \max \{ \sup_{x^* \in C_\Phi} \|x^*\|, \sup_{x^* \in C_v} \|x^*\| \} \\ &= \max\{|C_\Phi|, |C_v|\}. \end{aligned}$$

Let us first estimate

$$|C_\Phi| = \sup_{x^* \in C_\Phi} \|x^*\| = \sup_{\substack{S \in C_2 \\ z^* \in B_{Z^*}}} \|(\Phi(S))^*(z^*)\| = \sup_{S \in C_2} \|(\Phi(S))^*\| = \sup_{S \in C_2} \|\Phi(S)\|.$$

Using the conclusion of Lemma 1, we have for all $S \in C_2$,

$$\|S\| \leq \|\Phi(S)\| \leq \sup_{S \in C_2} \|\Phi(S)\| = |C_\Phi|$$

and

$$\|\Phi(S)\| \leq |C_2|.$$

Hence

$$|C_2| \leq |C_\Phi| \leq |C_2|,$$

meaning that $|C_\Phi| = |C_2|$. Let us now compute

$$|C_v| = \sup_{x^* \in C_v} \|x^*\| = \sup_{z \in B_Z} \|vz\| = \|v\| = 1.$$

Consequently,

$$|C| = \max \{|C_\Phi|,\ |C_v|\} = \max \{|C_2|,\ 1\}.$$

Assume that the result is true for $n-1$, where $n \in \{3, 4, \ldots\}$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}({}^{n-1}X))$. By Lemma 1 there exist a reflexive Banach space Z, a linear mapping $\Phi : \operatorname{span} C_n \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, \mathcal{L}^s_{wu}(^{n-1}X))$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} C_n$. Then for all $S \in C_n$ and for all $x \in X$, considering $(x, \ldots, x) \in (\mathcal{L}^s_{wu}(^{n-1}X))^*$ (note that if $A \in \mathcal{L}^s_{wu}(^{n-1}X)$, then $\langle (x, \ldots, x), A\rangle = A(x, \ldots, x)$),

$$|(Sx)(x, \ldots, x)| = |v(\Phi(S)x)(x, \ldots, x)| = |(v^*(x, \ldots, x))(\Phi(S)x)|,$$

hence

$$|(Sx)(x, \ldots, x)| \le \|v^*(x, \ldots, x)\| \|\Phi(S)x\|.$$

Put, as above,

$$C_\Phi = \overline{\{(\Phi(S))^*(z^*) : S \in C_n, z^* \in B_{Z^*}\}} \subset X^*.$$

Then C_Φ is circled and compact, and we get

$$\|\Phi(S)x\| = \sup_{z^* \in B_{Z^*}} |z^*(\Phi(S)x)| = \sup_{z^* \in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \le \sup_{x^* \in C_\Phi} |x^*(x)|$$

for all $S \in C_n$ and for all $x \in X$. Recall that $v(B_Z)$ is a relatively compact subset of $\mathcal{L}^s_{wu}(^{n-1}X)$. Hence

$$C_{n-1} := \{T_P : \ P \in \mathcal{P}_{wu}(^{n-1}X), \ A_P \in v(B_Z)\} \subset \mathcal{L}(X, \mathcal{L}^s(^{n-2}X))$$

is also relatively compact. According to [1], $C_{n-1} \subset \mathcal{K}(X, \mathcal{L}^s(^{n-2}X))$. Therefore, by the induction hypothesis, there is a circled and compact subset $C_v \subset X^*$ with $|C_v| = \max\{|C_{n-1}|, 1\}$ such that

$$|(T_P x)(x, \ldots, x)| \le \sup_{x^* \in \, C_v} |x^*(x)|^{n-1}$$

for all $P \in \mathcal{P}_{wu}(^{n-1}X)$ with $A_P \in v(B_Z)$. Since $v(B_Z) \subset \mathcal{L}^s_{wu}(^{n-1}X)$, for all $z \in B_Z$ there exists $P \in \mathcal{P}_{wu}(^{n-1}X)$ such that $vz = A_P$. By definition, $A_P(x, x, \ldots, x) = (T_P x)(x, \ldots, x)$, $x \in X$. Hence, for all $z \in B_Z$ and all $x \in X$,

$$|(vz)(x, \ldots, x)| = |A_P(x, x, \ldots, x)| = |(T_P x)(x, \ldots, x)| \le \sup_{x^* \in \, C_v} |x^*(x)|^{n-1}.$$

Therefore

$$\begin{aligned} \|v^*(x, \ldots, x)\| &= \sup_{z \in B_Z} |(v^*(x, \ldots, x))(z)| \\ &= \sup_{z \in B_Z} |(vz)(x, \ldots, x)| \le \sup_{x^* \in C_v} |x^*(x)|^{n-1}. \end{aligned}$$

Finally, let $C = C_\Phi \cup C_v$. Then C is circled and compact, and

$$\begin{aligned} |(Sx)(x, \ldots, x)| &\le \|v^*(x, \ldots, x)\| \|\Phi(S)x\| \\ &\le \sup_{x^* \in C_v} |x^*(x)|^{n-1} \sup_{x^* \in C_\Phi} |x^*(x)| \le \sup_{x^* \in C} |x^*(x)|^n \end{aligned}$$

for all $S \in C_n$ and all $x \in X$.

To complete the proof, let us show that $|C| = \max\{|C_n|,\ 1\}$. Similarly to the case $n = 2$, we have

$$|C| = \sup_{x^*\in C} \|x^*\| = \sup_{x^*\in C_\Phi\cup C_v} \|x^*\| = \max\{\sup_{x^*\in C_\Phi} \|x^*\|,\ \sup_{x^*\in C_v} \|x^*\|\}$$
$$= \max\{|C_\Phi|, |C_v|\}$$

and

$$|C_\Phi| = \sup_{x^*\in C_\Phi} \|x^*\| = \sup_{\substack{S\in C_n\\ z^*\in B_{Z^*}}} \|(\Phi(S))^*(z^*)\| = \sup_{S\in C_n} \|(\Phi(S))^*\| = \sup_{S\in C_n} \|\Phi(S)\|.$$

Using the conclusion of Lemma 1, we have for all $S \in C_n$,

$$\|S\| \leq \|\Phi(S)\| \leq |C_\Phi|$$

and

$$\|\Phi(S)\| \leq |C_n|.$$

Hence

$$|C_n| \leq |C_\Phi| \leq |C_n|,$$

meaning that $|C_\Phi| = |C_n|$. Let us show that $|C_v| = 1$. Recall that $|C_v| = \max\{|C_{n-1}|, 1\}$. Since

$$|C_{n-1}| = \sup_{T_P\in C_{n-1}} \|T_P\| = \sup_{A_P\in v(B_Z)} \|A_P\| \leq \sup_{z\in B_Z} \|vz\| = \|v\| = 1,$$

we clearly have $|C_v| = 1$. □

3. APPLICATION TO POLYNOMIALS

The next theorem is proved by Toma [4] (an alternative proof is given in [2]).

Theorem 3 [4]. *Let X be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}(^nX)$. The polynomial $P \in \mathcal{P}_{wu}(^nX)$ if and only if there exists a compact subset C of X^* such that for all $x \in X$*

$$|P(x)| \leq \sup_{x^*\in C} |x^*(x)|^n.$$

The following is a quantitative version of Theorem 3.

Corollary 1. *Let X be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}(^nX)$. The following are equivalent*:

(a) $P \in \mathcal{P}_{wu}(^nX)$,

(b) *there exists a compact subset C of X^* such that for all $x \in X$*

$$|P(x)| \leq \sup_{x^*\in C} |x^*(x)|^n,$$

(c) *there exists a compact circled subset C of X^* with*

$$\max\{\|P\|, 1\} \le |C| \le \max\left\{\frac{n^n}{n!}\|P\|, 1\right\}$$

such that for all $x \in X$

$$|P(x)| \le \sup_{x^* \in C} |x^*(x)|^n.$$

Proof. (a) $\Rightarrow$ (c). Let $P \in \mathcal{P}_{wu}(^nX)$, then $\{T_P\} \subset \mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$. Applying Theorem 2 to $C_n = \{T_P\}$, we get that there is a compact circled subset C of X^* with $|C| = \max\{\|T_p\|, 1\}$ such that for all $x \in X$

$$|P(x)| = |A_P(x, x, \ldots, x)| = |(T_P x)(x, \ldots, x)| \le \sup_{x^* \in C} |x^*(x)|^n.$$

Applying the polarization formula (see, for example, [5], Theorem 1.7), we have

$$\|P\| \le \|T_P\| \le \frac{n^n}{n!}\|P\|.$$

Hence $\max\{\|P\|, 1\} \le |C| \le \max\{\frac{n^n}{n!}\|P\|, 1\}$.

$(c) \Rightarrow (b)$. Obvious.

$(b) \Rightarrow (a)$. Follows immediately from Theorem 3. □

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Banachi ruumi ühikkeral nõrgalt ühtlaselt pidevatest polünoomidest

Kristel Mikkor

On tõestatud Aroni-Lindströmi-Ruessi-Ryani [2] ja Toma [4] teoreemide kvantitatiivsed versioonid Banachi ruumi ühikkeral nõrgalt ühtlaselt pidevate polünoomide kohta. Tõestusmeetod tugineb kompaktsete operaatorite kompaktsete hulkade ühtlasele faktorisatsioonile.