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Interpolation of approximation spaces with nonlinear projectors

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Abstract. Approximation spaces defined by multiparametric approximation families with possible nonlinear projectors are considered. It is shown that a real interpolation space for a tuple of such spaces is again an approximation space of the same type.

Key words: interpolation functor, approximation space, K-functional.

Let $\overrightarrow{X} = (X_0, X_1, ..., X_n)$ be a tuple of Banach (or quasi-Banach) spaces, i.e. each space X_i, $i = 0, 1, ..., n$, is a Banach (or quasi-Banach) space linearly and continuously embedded in some linear topological space $\mathcal{X}$. As usual, the interpolation space $K_{\overrightarrow{\theta},q}(\overrightarrow{X})$ is defined by the norm

$$\|x\|_{\overrightarrow{\theta},q} = \left(\int\limits_{\mathbb{R}_+^n} (t_1^{-\theta_1} ... t_n^{-\theta_n} K(\overrightarrow{t}, x, \overrightarrow{X}))^q \frac{\mathrm{d}t_1}{t_1} ... \frac{\mathrm{d}t_n}{t_n} \right)^{1/q},$$

where $\overrightarrow{\theta} = (\theta_0, \theta_1, ..., \theta_n)$, $0 < \theta_i < 1$, $\theta_0 + \theta_1 + ... + \theta_n = 1$, $\overrightarrow{t} = (t_1, ..., t_n) \in \mathbb{R}_+^n$ and

$$K(\overrightarrow{t}, x, \overrightarrow{X}) = \inf_{x = x_0 + ... + x_n} \left(\|x_0\|_{X_0} + t_1 \|x_1\|_{X_1} + ... + t_n \|x_n\|_{X_n} \right)$$

is the K-functional of the tuple $\overrightarrow{X}$.

Let $X \subset \mathcal{X}$ be a Banach space and $\mathcal{A} = \left\{ A_{\vec{m}} \subset \mathcal{X}, \vec{m} \in \mathbb{Z}_+^d \right\}$ be a family of linear subspaces $A_{\vec{m}}$, where $\vec{m} = (m_1, ..., m_d)$ is a d-dimensional index with non-negative coordinates $m_i \geq 0$. We assume that the index set is ordered in coordinatewise order, i.e. $\vec{m} \leq \vec{l}$ means that $m_i \leq l_i$ for $1 \leq i \leq d$.

Definition 1. *We will say that $(X, \mathcal{A})$ is a d-parametric approximation family if $\{0\} = A_{\vec{0}} \subset A_{\vec{m}} \subset A_{\vec{l}}$ for $\vec{m} \leq \vec{l}$.*

As usual, the approximation number $e_{\vec{k}}(x, X)$ for $x \in X$ is defined by the formula

$$e_{\vec{k}}(x, X) = \inf \left\{ \|x - a\|_X \, , \; a \in A_{\vec{k}} \cap X \right\}.$$

Let Φ be an ideal Banach space of functions $f : \mathbb{Z}_+^d \to \mathbb{R}$ such that

$$l_0(\mathbb{Z}_+^d) \subset \Phi \subset l_\infty(\mathbb{Z}_+^d),$$

where $l_0(\mathbb{Z}_+^d)$ is a space of functions with finite support.

Definition 2. *The approximation space $E_\Phi(X, \mathcal{A})$ is defined by the norm*

$$\|x\|_{E_\Phi(X,\mathcal{A})} = \left\| \left\{ e_{\vec{k}}(x, X) \right\}_{\vec{k} \in \mathbb{Z}_+^d} \right\|_\Phi .$$

Note that one-parametric approximation spaces have been considered by many authors (see, e.g., [1–5]). In the paper [6] multiparametric approximation spaces were considered, and conditions (on an interpolation functor $\mathcal{F}$ and approximation family $\mathcal{A}$) were given under which the interpolation space of a tuple $E_{\overrightarrow{\Phi}}(\overrightarrow{X}, \mathcal{A}) = (E_{\Phi_0}(X_0, \mathcal{A}), ..., E_{\Phi_n}(X_n, \mathcal{A}))$ is again the approximation space of the same type, i.e.,

$$\mathcal{F}[E_{\overrightarrow{\Phi}}(\overrightarrow{X}, \mathcal{A})] = E_{\mathcal{F}[\overrightarrow{\Phi}]}(\mathcal{F}[\overrightarrow{X}], \mathcal{A}). \tag{1}$$

A natural condition on the interpolation functor that arises here is the so-called splitting condition, namely

$$\mathcal{F}[\overrightarrow{\Phi}(\overrightarrow{X})] = \mathcal{F}[\overrightarrow{\Phi}](\mathcal{F}[(\overrightarrow{X})]), \tag{2}$$

where $\overrightarrow{\Phi}(\overrightarrow{X}) = (\Phi_0(X_0), ..., \Phi_n(X_n))$ is a tuple of vector-valued spaces $\Phi_i(X_i)$.

It is known that the "splitting condition" is not always fulfilled. The case where $\mathcal{F}$ is a functor of real interpolation $\mathcal{K}_{\overrightarrow{\theta},q}$ and $\overrightarrow{\Phi} = (l_{q_0}^{\vec{s}_0}, ..., l_{q_n}^{\vec{s}_n})$ is studied in [7] and [8]; this case is important for applications.

In [6] it was shown that the formula (1) holds for an interpolation functor $\mathcal{F}$ satisfying the "splitting condition" (2) and for a multiparametric approximation family $\mathcal{A}$ with some family of linear projectors. But in some cases, for example, when considering quasi-Banach spaces (see [9]), it is useful to have an analogous result for approximation families with nonlinear projectors.

Let us have d one-parametric approximation families

$$\mathcal{A}^{(k)} = \left\{ A_m^{(k)} \subset X_0 + ... + X_n, m \in \mathbb{Z}_+ \right\}, \quad k = 1, ..., d,$$

and let us consider a special d-parametric approximation family

$$\mathcal{A} = \left\{ A_{\overrightarrow{m}} = A_{m_1}^{(1)} + ... + A_{m_d}^{(d)} \right\}.$$

Definition 3. *We will say that* $(\overrightarrow{X}, \mathcal{A})$ *is complemented if there exists a family of (possibly nonlinear) operators* $P_m^{(k)} : X_0 + ... + X_n \to A_m^{(k)}$ *such that*

1. $P_m^{(k)} x = x$ *if* $x \in A_m^{(k)}$,

2. $P_{m_0}^{(k_0)} P_{m_1}^{(k_1)} = P_{m_1}^{(k_1)} P_{m_0}^{(k_0)}$,

3. $\left\| P_m^{(k)} x \right\|_{X_j} \leq \gamma \left\| x \right\|_{X_j}$ *with* γ *independent of* m, k, j, *and* x.

To formulate our first result, let us consider operators $Q_{\overrightarrow{m}} : X_0 + ... + X_n \to A_{\overrightarrow{m}}$ given by the formula

$$Q_{\overrightarrow{m}} = I - \prod_{i=1}^{d} (I - P_{m_i}^{(i)})$$

and let us also define operators

$$\triangle Q_{\overrightarrow{m}} = \prod_{i=1}^{d} (Q_{\overrightarrow{m}+e_i} - Q_{\overrightarrow{m}}),$$

where e_i, $1 \leq i \leq d$, is the standard basis in $\mathbb{R}^d$. Let $\overrightarrow{\Phi} = (\Phi_0, ..., \Phi_n)$ be a tuple of ideal spaces Φ_i with the Fatou property

$$\left\| \lim_{n \to \infty} f_n \right\|_{\Phi_i} \leq \underline{\lim}_{n \to \infty} \left\| f_n \right\|_{\Phi_i}$$

and such that the operator S is bounded in each Φ_i:

$$(Sf)(\vec{k}) = \sum_{\vec{l} \geq \vec{k}} f(\vec{l}), \vec{k} \in \mathbb{Z}_+^d.$$

Then the following theorem is true.

Theorem 4. *Suppose that* $(\overrightarrow{X}, \mathcal{A})$ *is complemented, the operators* $P_m^{(k)}$ *are linear for* $k \leq d-1$ *and the operators* $P_m^{(d)}$ *possess the following property: for any decomposition* $x = x_0 + ... + x_n$ $(x_j \in X_j)$ *there exists a decomposition* $P_m^{(d)} x = y_0^m + ... + y_n^m$ *such that*

$$\left\| x_j - y_j^m \right\|_{X_j} \leq \gamma e_m(x_j; \mathcal{A}, X_j),$$

where $\gamma > 0$ *is some constant independent of* x *and* m. *Then*

$$K(\cdot, x; E_{\overrightarrow{\Phi}}(\overrightarrow{X}, \mathcal{A})) \approx K(\cdot, \{\triangle Q_{\vec{m}} x\}_{\vec{m}}; \overrightarrow{\Phi}(\overrightarrow{X})).$$

The next theorem shows that spaces considered above are stable under real interpolation.

Theorem 5. *Suppose that the tuples* $\vec{\Phi}, \vec{X}$ *are such that for the interpolation functor* $K_{\vec{\theta},q}$ *the "splitting condition" is fulfilled. Then if the conditions of Theorem* 1 *hold*, *we have the equality*

$$K_{\vec{\theta},q}(E_{\vec{\Phi}}(\vec{X}, \mathcal{A})) = E_{K_{\vec{\theta},q}(\vec{\Phi})}(K_{\vec{\theta},q}(\vec{X}), \mathcal{A}).$$

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Aproksimatsiooniruumid mittelineaarsete projektoritega ja nende interpolatsioon

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On vaadeldud mitmest parameetrist sõltuvate parvede võimalike mittelineaarsete projektorite poolt defineeritud aproksimatsiooniruume. On näidatud, et selliste ruumide iga reaalne interpolatsiooniruum moodustab jälle sama tüüpi aproksimatsiooniruumi.