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μ-faster convergence and μ-acceleration of convergence by regular matrices

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Abstract. A new, nonclassical convergence acceleration concept, called μ-acceleration of convergence (where μ is a positive monotonically increasing sequence), is introduced and compared with the classical convergence acceleration concept. Regular matrix methods are used to accelerate the convergence of sequences. Kornfeld (*J. Comput. Appl. Math.*, 1994, **53**, 309–321) proved that if B-transform of every convergent sequence x converges not slower than its A-transform, where A and B are regular matrix methods, then A and B are equivalent. In this paper it is proved that Kornfeld's assertion cannot be transferred to μ-acceleration of convergence in a general case.

Key words: convergence acceleration, matrix methods, speed of convergence.

1. INTRODUCTION AND PRELIMINARIES

Problems connected with convergence of iterative sequences (for example, using iterative methods for solving equations and systems of equations, employing methods involving series expansions) often arise in numerical analysis. In many cases convergence of these sequences is slow. Therefore it is useful to apply convergence acceleration methods, which transform a slowly converging sequence $x = (x_n)$ into a new sequence $y = (y_n)$, converging to the same limit faster than the initial sequence. Throughout this paper we assume that indices and summation indices are integers, changing from 0 to ∞, if not specified otherwise. All notions not defined in this paper can be found in [1]. In the present paper regular matrix methods are used to accelerate the convergence of sequences.

Different methods are used to accelerate the convergence of a sequence and to estimate and compare the speeds of convergence of sequences (see, for example, [2–7]). Classically the convergence acceleration and the weak convergence acceleration of a sequence x are determined by the following definitions (cf. [2,5]).

Definition 1.1. *Let $x = (x_k)$ and $y = (y_k)$ be sequences with limits ς and ξ, respectively. If*

$$\lim_n \frac{|y_n - \xi|}{|x_n - \varsigma|} = 0, \tag{1.1}$$

then it is said that y converges faster than x.

Definition 1.2. *Let $x = (x_k)$ and $y = (y_k)$ be sequences with limits ς and ξ, respectively. It is said that y converges weakly faster than x if there exists a constant $K = K(x)$ such that*

$$|y_n - \xi| \leq K\,|x_n - \varsigma| \text{ for all } n. \tag{1.2}$$

Definition 1.3. *The sequence transformation $T : x \to y$ is said to*
(a) *accelerate the convergence of the sequence x if y converges faster than x,*
(b) *weakly accelerate the convergence of the sequence x if y converges weakly faster than x.*

A short overview of the research of convergence acceleration during the 20th century is given in [2]. From [2] we can conclude that in recent years the most significant results have been achieved using nonlinear methods of acceleration, but some reliable results have also been obtained with the help of linear methods (see, for example, [5,8–11]).

The convergence acceleration by matrix methods was studied in [5–11]. Let $A = (a_{nk})$ be a matrix with real or complex entries. A sequence $x = (x_k)$ is said to be A-*summable* if the sequence $Ax = (A_n x)$ is convergent, where

$$A_n x = \sum_k a_{nk} x_k.$$

We denote the set of all A-summable sequences by c_A. Thus, a matrix A determines the summability method on c_A, which we also denote by A. A method A is said to be *regular* if for each $x = (x_n) \in c$, where c is the set of convergent sequences, the equality $\lim_n A_n x = \lim_n x_n$ holds.

Definition 1.4 ([5], p. 310). *A regular matrix method A is said to be universally accelerating if $(A_n x)$ converges faster than x for every $x \in c$, and weakly accelerating if $(A_n x)$ converges weakly faster than x for every $x \in c$.*

Let A^p be the matrix obtained from A by crossing out the first $p+1$ rows of A. Kornfeld (see [5], pp. 311–320) proved the following

Theorem 1.1. *Let A and B be two regular methods. If for every $x \in c$ the sequence Bx converges weakly faster than Ax, then $A^p = B^p$ for some p.*

In a special case, where A is the identity method, i.e. $A = I = (\delta_{nk})$ with $\delta_{nn} = 1$ and $\delta_{nk} = 0$ for $n \neq k$, the next result follows from Theorem 1.1.

Corollary 1.1. *If a regular method $B = (b_{nk})$ is weakly accelerating, then there exists a number n_0 such that $b_{nk} = \delta_{nk}$ for all $n > n_0$, i.e. B is equivalent to identity method I.*

If for methods $A = (a_{nk})$ and $B = (b_{nk})$ for infinitely many n there exists $k = k(n)$ so that $a_{nk} \neq b_{nk}$, then we say that A *is essentially different from* B. From Corollary 1.1 we can conclude that any regular method, essentially different from I, cannot be weakly accelerating. As each universally accelerating method is weakly accelerating, any regular method, essentially different from I, cannot be universally accelerating.

Kangro [6,7] introduced the concepts of boundedness with speed and convergence with speed for estimating the speed of convergence of a sequence, and the concepts of A-boundedness with speed and A-summability with speed for accelerating the convergence of a sequence. Later these concepts were generalized and used for the acceleration of convergence by Tammeraid [8–11].

In the present paper matrix methods are used, but instead of the classical concept of convergence acceleration, a new concept of convergence acceleration is introduced. In comparison with Kangro's concepts of boundedness with speed and A-boundedness with speed, some new details are added. In Section 2 the concepts of *μ-faster convergence* (Definitions 2.1 and 2.1$'$) and weakly *μ-faster convergence* (Definition 2.4) are defined and compared with usual faster convergence and weakly faster convergence concepts, determined by Definitions 1.1 and 1.2, respectively. It is shown that the new concepts allow a more precise comparison of the speeds of convergence for a larger set of sequences than the classical concepts. In Example 2.1, with the help of Aitken's process (see [2], p. 4), the sequence is found, which converges μ-faster (but not faster) than an initial sequence. It is also proved that if for a sequence $x = (x_n)$ with the limit ς the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing, then the μ-faster as well as weakly μ-faster convergence of a sequence y with respect to x coincide with the usual weakly faster convergence of y with respect to x. In Section 3 the concepts of μ-acceleration of convergence (Definition 3.1) and weakly μ-acceleration of convergence (Definition 3.2) are defined. It is shown that the assertions of Theorem 1.1 and Corollary 1.1 cannot be transferred to weakly μ-acceleration of convergence in a general case. It means that there exists a regular matrix method which is weakly μ-accelerating with respect to another regular matrix method, and a regular matrix method, essentially different from I, can weakly μ-accelerate the convergence. However, it is proved that the assertion of Corollary 1.1 holds for weak μ-acceleration of convergence in the special case where $A = I$ and B is a *triangular matrix method*, i.e., when $b_{nk} = 0$ for $k > n$. It is also shown that under suitable assumptions the assertion of Theorem 1.1 holds for weak μ-acceleration of convergence in the case, where B is a triangular matrix method and A is a *normal matrix method*, i.e., when A is triangular and $a_{nn} \neq 0$ for all n.

2. ON μ-FASTER CONVERGENCE OF SEQUENCES

Here we introduce a new concept for comparison of speeds of convergence of sequences.

Definition 2.1. *Let $x = (x_n)$ and $y = (y_n)$ be convergent sequences with limits ς and ξ, respectively. We say that y converges μ-faster than x if*
(a) *there exists $\mu = \mu(x) = (\mu_n)$, $0 < \mu_n \nearrow \infty$, so that*

$$l_n = \mu_n |x_n - \varsigma| \neq O(1) \ \ and \ \ L_n = \mu_n |y_n - \xi| = O(1), \tag{2.1}$$

(b) *there does not exist $\mu = \mu(y) = (\mu_n)$, $0 < \mu_n \nearrow \infty$, with the properties*

$$l_n = O(1) \ \ and \ \ L_n \neq O(1). \tag{2.2}$$

Remark 2.1. If $l_n = O(1)$, then $x = (x_n)$ is said to be μ-bounded by Kangro [7].

Let

$$\varphi = \{x = (x_k) \mid x_k = \text{const, if } k > k_0\}$$

for some $k_0 = 0, 1, ...$. It is easy to see that for all μ we have $l_k = o(1)$ for each $x \in \varphi$.

Proposition 2.1. *For every sequence $x = (x_n) \in c \setminus \varphi$ with limit ς there exists $\mu = \mu(x) = (\mu_n)$, $0 < \mu_n \nearrow \infty$, such that $l_n = O(1)$ and $l_n \neq o(1)$.*
Proof. Every $x = (x_n)$ with limit ς can be represented in the form

$$x = x^0 + \varsigma e; \ \ x^0 = \left(x_n^0\right) \in c_0, \ \ e = (1, 1, ...), \tag{2.3}$$

where c_0 is the set of sequences converging to zero. Hence, for the proof it is sufficient to show that the assertion of Proposition 2.1 holds for all $x^0 \in c_0 \setminus \varphi$. For a given sequence $x^0 = \left(x_n^0\right) \in c_0 \setminus \varphi$ we form the subsequence $\left(x_{k_n}^0\right)$ of $\left(x_n^0\right)$, satisfying the properties

$$\left|x_{k_0}^0\right| = \max_{0 \leq i \leq \infty} \left|x_i^0\right|, \ \left|x_{k_{n+1}}^0\right| = \max_{i > k_n} \left|x_i^0\right|.$$

Obviously the sequence $\left(\left|x_{k_n}^0\right|\right)$ is monotonically decreasing.
Defining now $\mu = \mu(x) = (\mu_i)$ by the relation

$$\mu_i = \begin{cases} \frac{1}{\left|x_{k_0}^0\right|} & (i \leq k_0), \\ \frac{1}{\left|x_{k_{n+1}}^0\right|} & (k_n < i \leq k_{n+1}), \end{cases} \tag{2.4}$$

we notice that $l_n = O(1)$ and $l_n \neq o(1)$ for $\varsigma = 0$.

Let us study the relationship between the concepts of the classical faster convergence, determined by Definition 1.1, and μ-faster convergence. First we notice that there exists no convergent sequence, converging faster or μ-faster than any sequence of φ.

Proposition 2.2. *If a sequence $y = (y_n) \in c$ converges faster than $x = (x_n) \in c \setminus \varphi$, then y converges also μ-faster than x.*

Proof. For $y \in \varphi$ the assertion of Proposition 2.2 is clearly true. Thus, suppose that $y \in c\setminus\varphi$ converges faster than $x \in c\setminus\varphi$, i.e. relation (1.1) holds, and show that then y converges also μ-faster than x. By Proposition 2.1 there exists $\mu = \mu(x) = (\mu_n)$, $0 < \mu_n \nearrow \infty$, so that $l_n = O(1)$ and $l_n \neq o(1)$. Using relation (1.1), we have now

$$\lim_n \frac{\mu_n |y_n - \xi|}{\mu_n |x_n - \varsigma|} = 0. \tag{2.5}$$

Consequently, by Proposition 2.1 there exists $\lambda = (\lambda_n)$, $0 < \lambda_n \nearrow \infty$, so that

$$\lambda_n \frac{\mu_n |y_n - \xi|}{\mu_n |x_n - \varsigma|} = O(1).$$

Denoting $\lambda_n \mu_n = \vartheta_n$, we get from the last relation that $\vartheta_n |y_n - \xi| = O(1)$ with $0 < \vartheta_n \nearrow \infty$. Moreover, $\vartheta_n |x_n - \varsigma| \neq O(1)$. Indeed, the relations $l_n = O(1)$ and $l_n \neq o(1)$ imply that there exists a subsequence (μ_{k_n}) of (μ_n) so that

$$\mu_{k_n} |x_{k_n} - \varsigma| \geq m$$

for some $m > 0$. Consequently,

$$\vartheta_{k_n} |x_{k_n} - \varsigma| \neq O(1).$$

From equality (2.5) we see that there exists no μ with $l_n = O(1)$ and $L_n \neq O(1)$. Thus y converges μ-faster than x by Definition 2.1.

The converse assertion to Proposition 2.2, however, is not valid.

Example 2.1. Let $x = (x_n) \in c \setminus \varphi$ be given by the relations

$$x_n = \frac{1}{(n+1)2^n} \text{ if } n = 3k, \tag{2.6}$$

$$(n+1)^3 2^n x_n = o(1) \text{ if } n = 3k+1, \tag{2.7}$$

and

$$(n+1)^2 2^n x_n \neq O(1),\ 2^n x_n = o(1) \text{ if } n = 3k+2, \tag{2.8}$$

where $k = 0, 1, ...$, i.e. the subsequence (x_{3k}) of the sequence (x_n) is given exactly, but for the subsequences (x_{3k+1}) and (x_{3k+2}) of (x_n) only the estimations (2.7) and (2.8) are given. The sequence transformation

$$y_n = x_{3n} - \frac{(x_{3n+3} - x_{3n})^2}{x_{3n+6} - 2x_{3n+3} + x_{3n}} \tag{2.9}$$

we may consider as Aitken's process (see [2], p. 4) applied to the subsequence (x_{3k}) of x. Using (2.6), we get from (2.9) that

$$y_n = \frac{9}{2^{3n}\,(1323n^3 + 6993n^2 + 12024n + 6736)}.$$

It is easy to see that now $\xi = \varsigma = 0$ and $y = (y_n)$ converges not faster than x and x converges not faster than y, but y converges μ-faster than x by Definition 2.1. Indeed, relations (2.1) hold for $\mu = (\mu_n)$, defined by the equalities $\mu_n = 2^{3n}(n+1)^3$, but there does not exist $\mu = \mu(y) = (\mu_n)$, $0 < \mu_n \nearrow \infty$, with properties (2.2).

Thus, by Proposition 2.2 and Example 2.1 we can say that the μ-faster convergence concept, determined by Definition 2.1, allows us to compare the speeds of convergence for a larger set of sequences than the classical faster convergence concept determined by Definition 1.1.

Further, let us show that the new concept allows a more precise comparison of the convergence speeds of sequences than the classical concept. For every sequence $x \in c \setminus \varphi$ we denote

$$\mu_x = \{\mu = (\mu_n) \mid 0 < \mu_n \nearrow \infty,\ l_n = \mu_n \left| x_n - \lim_n x_n \right| = O(1),\ \ l_n \neq o(1)\}.$$

Definition 2.2. *We say that a sequence $\mu \in \mu_x$ is a speed of convergence of x and a sequence $\mu^* = (\mu_n^*) \in \mu_x$ is the limit speed of convergence of x if for all $\mu = (\mu_n) \in \mu_x$ the relation $\mu_n/\mu_n^* = O(1)$ holds.*

From the proof of Proposition 2.1 we see that the speed μ, defined by (2.4), is also the limit speed of $x \in c \setminus \varphi$, represented in form (2.3). Therefore, from Proposition 2.1 we get

Corollary 2.1. *Every sequence $x \in c \setminus \varphi$ has the limit speed of convergence.*

Remark 2.2. If for a sequence $\mu = (\mu_n) \in \mu_x$ the inequality $\mu_n/\mu_n^* > m$ holds for some $m > 0$, where μ^* is the limit speed of x, then μ is also the limit speed of x.

Remark 2.3. If for $x = (x_n) \in c \setminus \varphi$ the relation

$$m < \mu_n \left| x_n - \varsigma \right| < M$$

is valid for some $m > 0$ and $M > 0$, where ς is the limit of x and $\mu = (\mu_n) \in \mu_x$, then μ is the limit speed of x.

Remark 2.4. A sequence $x \in \varphi$ has neither speed nor limit speed in the sense of Definition 2.2.

Proposition 2.3. *For each $x \in c \setminus \varphi$ there exists an element $\mu \in \mu_x$, which is not the limit speed of x.*

Proof. Let $\mu^* = (\mu_n^*) \in \mu_x$ be the limit speed of x and x be represented in form (2.3). Then there exists a subsequence $\left(\mu_{k_n}^*\right)$ of (μ_n^*) so that

$$\mu_{k_n}^* \left|x_{k_n}^0\right| > m$$

for some $m > 0$ and $\mu_{k_n}^*/\mu_{k_{n+1}-1}^* \to 0$. We define a sequence $\mu = (\mu_i)$ as follows:

$$\mu_i = \begin{cases} \mu_i^* & (i \leq k_0), \\ \mu_{k_n}^* & (k_n \leq i < k_{n+1}). \end{cases}$$

Then

$$\frac{\mu_i}{\mu_i^*} = \begin{cases} 1 & (i \leq k_0 \text{ and } i = k_n), \\ \frac{\mu_{k_n}^*}{\mu_i^*} & (k_n < i < k_{n+1}). \end{cases}$$

Now we see that $\mu_n \left|x_n^0\right| = O(1)$ and $\mu_n \left|x_n^0\right| \neq o(1)$. Hence $\mu \in \mu_x$. However, for $i = k_{n+1} - 1$ we get

$$\frac{\mu_i}{\mu_i^*} = \frac{\mu_{k_n}^*}{\mu_{k_{n+1}-1}^*} \to 0 \text{ for } n \to \infty,$$

i.e. the sequence (μ_i/μ_i^*) is not lower-bounded. Therefore μ is not the limit speed of x.

Remark 2.5. Condition (b) in Definition 2.1 is essential. Let us explain the importance of condition (b) with the help of the following example.

Example 2.2. Let a sequence $x = (x_n)$ be defined by the equality

$$x_n = \frac{1}{(n+1)^{2n}}$$

and a sequence $y = (y_n)$ by the equalities

$$y_n = \begin{cases} x_n & (n = 2k), \\ \frac{1}{(n+1)^{2n-1}} & (n = 4k+1), \\ \frac{1}{(n+1)^{2n+1}} & (n = 4k+3), \end{cases}$$

where $k = 0, 1, \ldots$. The limit of both sequences is 0. We define the sequences $\mu = (\mu_n)$ and $\lambda = (\lambda_n)$ with the help of the equalities

$$\mu_n = (n+1)^{2n} \quad \text{and} \quad \lambda_n = \begin{cases} \mu_n & (n = 2k), \\ (n+1)^{2n-1} & (n = 4k+1), \\ (n+1)^{2n+1} & (n = 4k+3), \end{cases}$$

where $k = 0, 1, ...$. Then we see that $\mu_n |x_n| = O(1)$ and $\mu_n |y_n| \neq O(1)$, but at the same time $\lambda_n |y_n| = O(1)$ and $\mu_n |y_n| \neq O(1)$. It means that we cannot say that x converges μ-faster than y or y converges μ-faster than x.

Definition 2.3. *We say that the limit speed of convergence $\mu^* = (\mu_n^*)$ of a sequence y is higher than the limit speed of convergence $\lambda^* = (\lambda_n^*)$ of a sequence x if the ratio λ_n^*/μ_n^* is upper-bounded, but not lower-bounded.*

Using the concept of the limit speed of convergence, we can reformulate Definition 2.1.

Definition 2.1′. *We say that a sequence y converges μ-faster than x if the limit speed of convergence of y is higher than the limit speed of convergence of x or $y \in \varphi$ and x does not belong to φ.*

Definition 2.4. *We say that y converges weakly μ-faster than x if the limit speed of convergence of x is not higher than the limit speed of convergence of y or $y \in \varphi$.*

Of course, if a sequence $y \in c \setminus \varphi$ converges μ-faster than x, then y converges also weakly μ-faster than x.

Further, we study the relationship between weakly μ-faster convergence and usual weakly faster convergence.

Proposition 2.4. *If a sequence y converges weakly faster than x, then y converges weakly μ-faster than x.*

Proof. If $y \in \varphi$, then the assertion of Proposition 2.4 is clearly valid. We suppose $y \in c \setminus \varphi$. Then by relation (2.3) it is sufficient to prove that the assertion of Proposition 2.4 holds for all $x = (x_n) \in c_0$ and $y = (y_n) \in c_0$, for which y converges weakly faster than x. Thus, supposing that y converges weakly faster than x, we get by Definition 1.2 that the relation

$$|y_n| \leq K |x_n| \tag{2.10}$$

holds for all n and some number $K > 0$. By Corollary 2.1 the sequence x has the limit speed $\lambda^* = (\lambda_n^*)$. Hence, by relation (2.10) we get

$$\lambda_n^* |y_n| \leq K\lambda_n^* |x_n| = O(1),$$

i.e.

$$\lambda_n^* |y_n| = O(1).$$

Consequently, the limit speed of y cannot be lower than the limit speed of x. Thus y converges weakly μ-faster than x.

However, the converse assertion to the assertion of Proposition 2.4 is not valid. Indeed, in Example 2.1 the sequence y converges weakly μ-faster than x, but we cannot say that y converges weakly faster than x or that x converges weakly faster than y. Therefore, we can assert that, using the concept of weakly μ-faster convergence, it is possible to compare the speeds of convergence for a larger set of sequences and make it more exactly than with the help of the concept of usual weakly faster convergence, determined by Definition 1.2.

Let now $x = (x_n) \in c \backslash \varphi$ with the limit ς be a sequence for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. We show that in this case the μ-faster convergence as well as the weakly μ-faster convergence coincide with the usual weakly faster convergence.

Proposition 2.5. *Let $x = (x_n) \in c \setminus \varphi$ be a sequence with the limit ς, for which the sequence of absolute differences $(|x_n - \varsigma|)$ is monotonically decreasing. If a sequence $y = (y_n)$ converges μ-faster or weakly μ-faster than x, then y converges weakly faster than x.*

Proof. It is not difficult to see that the limit speed $\lambda^* = (\lambda^*_n)$ of a sequence x can be defined by the equality

$$\lambda^*_n = \frac{1}{|x_n - \varsigma|}.$$

Let $\mu^* = (\mu^*_n)$ be the limit speed of y. Then we get

$$\frac{\mu^*_n \, |y_n - \xi|}{\lambda^*_n \, |x_n - \varsigma|} = \mu^*_n \, |y_n - \xi| = O(1),$$

where ξ is the limit of y. The last relation implies inequality (1.2) for some constant K, because in this case $\mu^*_n/\lambda^*_n > m$ for some $m > 0$. Thus the assertion of Proposition 2.5 is valid.

Remark 2.6. It is easy to see that every subsequence of a convergent sequence x converges weakly μ-faster than x. Yet, it is not true for the concept of usual weakly faster convergence.

Example 2.3. Consider the sequence x, introduced in Example 2.1. It is incorrect to say that the subsequence (x_{2n}) of x converges weakly faster than x in the sense of Definition 1.2.

3. μ-ACCELERATION OF CONVERGENCE BY REGULAR MATRIX METHODS

We consider the convergence acceleration of sequences in the sense different from the classical concept.

Definition 3.1. *We say that a regular matrix method A μ-accelerates the convergence of a sequence $x \in c$ if the sequence Ax converges μ-faster than x.*

It is clear that μ-acceleration of all convergent sequences by a regular method A is not possible, because it is not possible to μ-accelerate the convergence of any $x \in \varphi$.

Definition 3.2. *We say that a regular method A weakly μ-accelerates the convergence of a sequence $x \in c$ if the sequence Ax converges weakly μ-faster than x. If the sequence Ax converges weakly μ-faster than x for all $x \in c$, then we say that A weakly μ-accelerates the convergence.*

From Proposition 2.4 we immediately get

Corollary 3.1. *If a regular method A weakly accelerates the convergence, then A also weakly μ-accelerates the convergence.*

Definition 3.3. *Let A and B be two regular methods with $c_A \subseteq c_B$. We say that B is weakly μ-accelerating with respect to A if for every $x \in c_A$ its B-transform Bx converges weakly μ-faster than its A-transform Ax.*

According to Kornfeld [5], *a regular method B is said to be weakly accelerating with respect to another regular method A* if for every $x \in c_A$ its B-transform Bx converges weakly faster than its A-transform Ax. From Proposition 2.4 we immediately get

Corollary 3.2. *Let A and B be two regular methods with $c_A \subseteq c_B$. If B is weakly accelerating with respect to A, then B is also weakly μ-accelerating with respect to A.*

We show that the converse assertions to Corollaries 3.1 and 3.2 are not valid. For this purpose we first show that the assertions of Theorem 1.1 and Corollary 1.1 are not valid for weak μ-acceleration of convergence in a general case. Let us present a counterexample.

Example 3.1. Let A be a regular method and a method B be defined by the relation

$$b_{nk} = a_{\rho(n),k},$$

where $\rho = \rho(n)$ is an integer-valued function satisfying the condition $\rho(n) \geq n$. Then B is a regular method (see Theorem 2.3.7 of [1]) and $(B_n x)$ is a subsequence of the sequence $(A_n x)$ for each $x \in c_A$. Therefore B is weakly μ-accelerating with respect to A by Remark 2.6. If $A = I$, then B weakly μ-accelerates the convergence.

Now it is easy to see that the converse assertions to Corollaries 3.1 and 3.2 do not hold. Indeed, let regular methods A and B be defined as in Example 3.1. Then for every $x \in c_A$ its B-transform Bx converges weakly μ-faster than its A-transform Ax. However, by Theorem 1.1 B-transform Bx of x converges weakly faster than A-transform Ax of x not for all sequences $x \in c_A$. For $A = I$ we get that Bx converges weakly faster than x not for all sequences $x \in c_A$.

Let now A be a normal regular matrix method and B a triangular regular matrix method. We show that in this case the assertion of Theorem 1.1 holds for weak μ-acceleration of convergence if $c_A \subseteq c_B$ and B is *consistent* with A, i.e.

$$\lim_n B_n x = \lim_n A_n x$$

for every $x \in c_A$.

Theorem 3.1. *Let A be a normal regular method. Let B be a triangular regular method with $c_A \subseteq c_B$ and B be consistent with A. If B is weakly μ-accelerating with respect to A, then $B^p = A^p$ for some p.*

Before proving Theorem 3.1 we prove this theorem in the special case where $A = I$, i.e. we show that for a triangular matrix method B the assertion of Corollary 1.1 can be transferred to weak μ-acceleration.

Theorem 3.2. *If a triangular regular method $B = (b_{nk})$ weakly μ-accelerates the convergence, then there exists a natural number n_0 so that $b_{nk} = \delta_{nk}$ for all $n > n_0$, i.e. B is equivalent to I.*

Proof. First we notice that a regular method B can weakly μ-accelerate the convergence only if the condition

$$\sum_{k=0}^{n} b_{nk} = 1 \tag{3.1}$$

holds for all n. Indeed, otherwise B cannot weakly μ-accelerate the convergence of all sequences of φ, for example, then the sequence e converges μ-faster than Be. Thus, let $B = (b_{nk})$ be a triangular regular method satisfying condition (3.1) and weakly μ-accelerating the convergence. Suppose, on the contrary, that $b_{nk} \neq \delta_{nk}$ for infinitely many n and show that in this case B cannot weakly μ-accelerate the convergence. To prove the last assertion, it is sufficient to construct such a sequence $x \in c_0$, which converges μ-faster than its B-transform $y = Bx$. Consider two different cases.

I. Assume that B has such a column b_{nk_0} (k_0 is fixed and $n = 0, 1, ...$), where infinitely many elements are different from zero. Let $b_{n_0k_0} \neq 0$ for some $n_0 > k_0$, $x_k = 0$ for $0 \leq k < k_0$ and $k_0 < k < n_0$. We choose x_{k_0}, x_{n_0} so that

$$0 < |x_{n_0}| < |x_{k_0}|,$$

$$\frac{|b_{n_0k_0}x_{k_0} + b_{n_0n_0}x_{n_0},|}{|x_{n_0}|} > n_0,$$

and

$$b_{nk_0}x_{k_0} + b_{nn_0}x_{n_0} \neq 0$$

for all $n > n_0$, where $b_{nk_0} \neq 0$ or $b_{nn_0} \neq 0$. We note that such numbers x_{k_0}, x_{n_0} exist, because the set of the existing ratios $\{-b_{nk_0}/b_{nn_0};\ n = 0, 1, ...\}$ is finite or countable. Now we choose the minimal number $n_1 > n_0$, for which $b_{n_1k_0} \neq 0$ or $b_{n_1n_0} \neq 0$. Then we can take $x_k = 0$ for $n_0 < k < n_1$ and choose x_{n_1} such that

$$0 < |x_{n_1}| < |x_{n_0}|,$$

$$\frac{|b_{n_1k_0}x_{k_0} + b_{n_1n_0}x_{n_0} + b_{n_1n_1}x_{n_1}|}{|x_{n_1}|} > n_1,$$

and

$$b_{nk_0}x_{k_0} + b_{nn_0}x_{n_0} + b_{nn_1}x_{n_1} \neq 0$$

for all $n > n_1$, where at least one of the numbers b_{nk_0}, b_{nn_0}, and b_{nn_1} is not zero. Continuing in a similar way, we will choose the sequence of natural numbers (n_i) $(n_0 < n_1 < ... < n_i < ...)$ and the numbers x_k so that

$$0 < ... < |x_{n_i}| < \left|x_{n_{i-1}}\right| < ... < |x_{n_0}| < |x_{k_0}|,$$

$$x_k = 0 \text{ for } k \neq k_0 \text{ and } k \neq n_i,$$

$$\frac{|b_{n_ik_0}x_{k_0} + b_{n_in_0}x_{n_0} + ... + b_{n_in_i}x_{n_i}|}{|x_{n_i}|} > n_i,$$

and

$$b_{nk_0}x_{k_0} + b_{nn_0}x_{n_0} + b_{nn_1}x_{n_1} + ... + b_{nn_i}x_{n_i} \neq 0$$

for all $n > n_i$, where at least one of the numbers b_{nk_0}, b_{nn_0}, b_{nn_1},..., b_{nn_i} is not equal to zero.

Thus we have constructed two sequences $x = (x_n)$ and $y = (y_n) = (B_n x)$ so that

$$y_n = x_n = 0, \text{ if } n \neq k_0 \text{ and } n \neq n_i, \; y_{k_0} = b_{k_0k_0}x_{k_0}$$

and

$$\frac{|y_{n_i}|}{|x_{n_i}|} > n_i. \tag{3.2}$$

We notice that nonzero elements of both sequences form the monotonically decreasing subsequences of these sequences. Therefore we can determine the limit speeds of convergence of x and y respectively by $\lambda^* = \left(\lambda_j^*\right)$ and $\mu^* = \left(\mu_j^*\right)$, where

$$\lambda_j^* = \begin{cases} \frac{1}{|x_{n_0}|} & (0 \leq j \leq n_0), \\ \frac{1}{|x_{n_i}|} & (n_{i-1} < j \leq n_i) \end{cases} \tag{3.3}$$

and

$$\mu_j^* = \begin{cases} \frac{1}{|y_{n_0}|} & (0 \leq j \leq n_0), \\ \frac{1}{|y_{n_i}|} & (n_{i-1} < j \leq n_i) \end{cases} \tag{3.4}$$

$(i = 1, 2, ...)$. By relation (3.2) we have $\lambda_j^*/\mu_j^* \to \infty$. Consequently, x converges μ-faster than its B-transform Bx and thus B cannot weakly μ-accelerate the convergence.

II. Assume that B is a matrix with finite columns, i.e. every column of B has a finite number of nonzero elements. In this case we also choose a number n_0, for which $b_{n_0k_0} \neq 0$ for some k_0 with $0 \leq k_0 < n_0$. Further, if possible, we continue as in case I. However, now it can happen that, after choosing n_i for some i, we have $b_{nk_0} = b_{nn_0} = b_{nn_1} = ... = b_{nn_i} = 0$ for every $n > n_i$. Thus, it is not possible to choose the next number n_{i+1}, as we did in case I. Therefore we proceed with the following

Step A. We choose a number $n'_i > n_i$ so that

$$b_{n'_i k_i} \neq 0 \text{ for } k_i \text{ with } n_i < k_i < n'_i. \tag{3.5}$$

Such n'_i exists. Indeed, there exists a number $\tilde{n}$ with $\tilde{n} > n_i$ so that $b_{nk} = 0$ for all $n > \tilde{n}$ and $k < n_i$, since B is the matrix with finite columns. As $b_{nk} \neq \delta_{nk}$ for infinitely many n and condition (3.1) is satisfied, we can take the minimal number $n'_i > n_i$, for which relation (3.5) holds. Further we choose x_{k_i} for $n_i < k_i < n'_i$ and $x_{n'_i}$ so that

$$0 < \left|x_{n'_i}\right| < |x_{k_i}| < |x_{n_i}|$$

and

$$\frac{\left|b_{n'_i k_i} x_{k_i} + b_{n'_i n'_i} x_{n'_i}\right|}{\left|x_{n'_i}\right|} > n'_i, \ \ i = 1, 2, \ldots .$$

If there exists $n > n'_i$ such that $b_{nk_i} \neq 0$ or $b_{nn'_i} \neq 0$, we can determine $x_{n_{i+1}}$ $(n_{i+1} > n'_i)$ as in case I. If not, we repeat Step A, choosing the next elements $x_{k_{i+1}}$ and $x_{n'_{i+1}}$ with

$$0 < \left|x_{n'_{i+1}}\right| < \left|x_{k_{i+1}}\right| < \left|x_{n'_i}\right| .$$

So we have constructed two sequences $x = (x_n)$ and $y = (y_n) = (B_n x)$, where

$$y_n = x_n = 0, \text{ if } n \neq k_i, \ n \neq n_i \ (i = 0, 1, \ldots),$$

$$y_{k_0} = b_{k_0 k_0} x_{k_0} \text{ and } y_{k_i} = x_{k_i} \ (i = 1, \ldots),$$

$$\left|y_{n'_i}\right| > n'_i \left|x_{n'_i}\right|, \tag{3.6}$$

and relation (3.2) holds. Consequently, if we have no necessity to use Step A, we can determine the limit speeds $\lambda^* = \left(\lambda^*_j\right)$ and $\mu^* = \left(\mu^*_j\right)$ of x and y by equalities (3.3) and (3.4), respectively. If we use Step A, we can determine the limit speeds of x and y also by equalities (3.3) and (3.4), replacing in them some elements λ^*_j by $1/\left|x_{n'_i}\right|$ and $1/|x_{k_i}|$ and some elements μ^*_j by $1/\left|y_{n'_i}\right|$ and $1/|y_{k_i}|$, respectively. Hence $\lambda^*_{k_i}/\mu^*_{k_i} = 1$, but by relations (3.2) and (3.6) we have $1 \leq \lambda^*_n/\mu^*_n \neq O(1)$. Consequently, x converges μ-faster than its B-transform y. Thus, again, we can conclude that B cannot weakly μ-accelerate the convergence.

Consequently, our assumption that $b_{nk} \neq \delta_{nk}$ for infinitely many n was incorrect and therefore there exists a number n_0 so that $b_{nk} = \delta_{nk}$ for all $n > n_0$.

Now we are able to prove Theorem 3.1.

The proof of Theorem 3.1. As A is a normal matrix method, we get for every $x \in c_A$ that

$$Bx = Cy; \;\; C = BA^{-1}, \; y = Ax, \tag{3.7}$$

where A^{-1} is the inverse matrix of A. On the other hand, for each $y \in c$ there exists a unique $x \in c_A$ so that relation (3.7) holds (see [1], p. 37). Consequently, $B = (b_{nk})$ can be weakly μ-accelerating with respect to A if and only if $C = (c_{nk})$ weakly μ-accelerates the convergence. In addition, the method C is triangular and regular, because $c_A \subseteq c_B$ and B is consistent with A (see [1], p. 76). Therefore, by Theorem 3.2 there exists a natural number p so that $c_{nk} = \delta_{nk}$ for $n > p$. This implies $b_{nk} = a_{nk}$ for $n > p$, since $B = CA$, i.e., $B^p = A^p$.

From Theorems 3.1 and 3.2, respectively, we immediately get the following corollaries.

Corollary 3.3. *Let A be a normal regular method, and B a triangular regular method, essentially different from A. Let $c_A \subseteq c_B$ and B be consistent with A. Then B cannot be weakly μ-accelerating with respect to A.*

Corollary 3.4. *Any triangular regular matrix, essentially different from I, cannot weakly μ-accelerate the convergence.*

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μ-kiiremast koonduvusest ja koonduvuse μ-kiirendamisest regulaarsete maatriksitega

Ants Aasma

On esitatud koonduvuse μ-kiirendamise mõiste (μ on positiivne monotoonselt kasvav jada) ja seda on võrreldud klassikalise koonduvuse kiirendamise mõistega. Koonduvuse μ-kiirendamist on uuritud regulaarsete maatriksmeetoditega. Kornfeld (*J. Comput. Appl. Math.*, 1994, **53**, 309–321) tõestas, et kui iga koonduva jada B-teisendus ei koondu aeglasemalt kui selle jada A-teisendus, kus A ja B on regulaarsed maatriksmeetodid, siis A ja B on ekvivalentsed. Artiklis on tõestatud, et üldjuhul ei saa Kornfeldi väidet koonduvuse μ-kiirendamisele üle kanda. Siiski on näidatud, et teatavatel eeldustel kehtib see väide ka koonduvuse μ-kiirendamise jaoks erijuhul, kui A on normaalne ja B on kolmnurkne maatriksmeetod.