Proc. Estonian Acad. Sci. Phys. Math., 2006, **55**, 4, 210–219

On a class of Lorentzian para-Sasakian manifolds

Cengizhan Murathan[a], Ahmet Yıldız[b], Kadri Arslan[a], and Uday Chand De[c]

[a] Department of Mathematics, Uludağ University, 16059 Bursa, Turkey; cengiz@uludag.edu.tr, arslan@uludag.edu.tr
[b] Department of Mathematics, Dumlupınar University, Kütahya, Turkey; ahmetyildiz@dumlupinar.edu.tr
[c] Department of Mathematics, Kalyani University, Kalyani, India; uc_de@yahoo.com

Received 23 March 2006

Abstract. We classify Lorentzian para-Sasakian manifolds which satisfy $P \cdot C = 0$, $Z \cdot C = L_C Q(g, C)$, $P \cdot Z - Z \cdot P = 0$, and $P \cdot Z + Z \cdot P = 0$, where P is the $v-$Weyl projective tensor, Z is the concircular tensor, and C is the Weyl conformal curvature tensor.

Key words: contact metric manifold, Lorentzian para-Sasakian manifold, Sasakian manifold, $v-$Weyl projective tensor, concircular tensor.

1. INTRODUCTION

Matsumoto [1] introduced the notion of Lorentzian para-Sasakian (*LP*-Sasakian for short) manifold. Mihai and Rosca defined the same notion independently in [2]. This type of manifold is also discussed in [3,4].

Let M be an n-dimensional Riemannian manifold of class C^∞. A v-projective symmetry is a projectable vector field X with the property in which every diffeomorphism φ of its one-parametric group is a projective map between leaves. In the theory of the projective transformations of connections the Weyl projective tensor plays an important role.

Recently, the authors of [5] studied the contact metric manifold M^n satisfying the curvature conditions $Z(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot Z = 0$, where Z is the *concircular tensor* of M^n defined by

$$Z(X, Y)W = R(X, Y)W - \frac{\tau}{n(n-1)} R_0(X, Y)W, \tag{1}$$

where

$$R_0(X, Y)W = g(Y, W)X - g(X, W)Y,$$

R and τ are the *Riemannian–Christoffel curvature tensor* and the *scalar curvature* of M^n, respectively. They observed immediately from the form of the concircular curvature tensor that Riemannian manifolds with a vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of the failure of a Riemannian manifold to be of constant curvature.

In the theory of the projective transformations of connections the Weyl projective tensor plays an important role. The $v-$*Weyl projective tensor* P in a Riemannian manifold (M^n, g) is defined by [6]

$$P(X, Y)W = R(X, Y)W - \frac{1}{n-1} R_1(X, Y)W, \tag{2}$$

where

$$R_1(X, Y)W = S(Y, W)X - S(X, W)Y,$$

with S being the *Ricci tensor* of M.

In the present study we give a classification of the *LP*-Sasakian manifold M^n satisfying the curvature conditions $P \cdot C = 0$, $Z \cdot C = L_C Q(g, C)$, $P \cdot Z - Z \cdot P = 0$, and $P \cdot Z + Z \cdot P = 0$, where Z is the concircular tensor defined by (1), P is the $v-$Weyl projective tensor, and C is the Weyl conformal curvature tensor of M^n.

2. PRELIMINARIES

A differentiable manifold of dimension n is called an *LP*-Sasakian manifold [1,2] if it admits a $(1, 1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η, and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{3}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{5}$$

$$g(X, \xi) = \eta(X), \qquad \nabla_X \xi = \phi X, \tag{6}$$

$$\Phi(X, Y) = g(X, \phi Y) = g(\phi X, Y) = \Phi(Y, X), \tag{7}$$

$$(\nabla_X \Phi)(Y, W) = g(Y, (\nabla_X \Phi)W) = (\nabla_X \Phi)(W, Y), \tag{8}$$

where ∇ is the covariant differentiation with respect to g. The Lorentzian metric g makes a timelike unit vector field, that is, $g(\xi, \xi) = -1$. The manifold M^n equipped with a Lorentzian almost paracontact structure (ϕ,ξ,η,g) is said to be a *Lorentzian almost paracontact manifold* (see [1,3]).

If we replace in (3) and (4) ξ by $-\xi$, then the triple (ϕ,ξ,η) is an almost paracontact structure on M^n defined by Sato [7]. The Lorentzian metric given by (6) stands analogous to the almost paracontact Riemannian metric for any almost paracontact manifold (see [7,8]).

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ,ξ,η,g) is called a *Lorentzian paracontact manifold* (see [1]) if

$$\Phi(X,Y) = \frac{1}{2}\left((\nabla_X \eta)Y + (\nabla_Y \eta)X\right).$$

A Lorentzian almost paracontact manifold M^n, equipped with the structure (ϕ,ξ,η,g), is called an *LP-Sasakian manifold* (see [1]) if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an *LP*-Sasakian manifold the 1-form η is closed. In [1] it is also proved that if an n-dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)W = g(X,Y)\eta(W) + g(X,W)\eta(Y) + 2\eta(X)\eta(Y)\eta(W),$$

then M^n admits an *LP*-Sasakian structure.

Further, on such an *LP*-Sasakian manifold M^n with the structure (ϕ,ξ,η,g) the following relations hold:

$$g(R(X,Y)W,\xi) = \eta(R(X,Y)W) = g(Y,W)\eta(X) - g(X,W)\eta(Y), \tag{9}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{11}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{12}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{13}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{14}$$

for any vector fields X, Y (see [1,2]), where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

An *LP*-Sasakian manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{15}$$

for any vector fields X, Y, where a, b are functions on M^n(see [9,10]).

Next we define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]}W, \tag{16}$$

$$(X \wedge_A Y)W = A(Y,W)X - A(X,W)Y, \tag{17}$$

respectively, where $X, Y, W \in \chi(M)$ and A is the symmetric $(0,2)$-tensor.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M,g) we define $P \cdot T$, $Z \cdot T$, and $Q(g,T)$ by

$$\begin{aligned}(P(X,Y)\cdot T)(X_1,...,X_k) = &- T(P(X,Y)X_1,X_2,...,X_k)\\ &- T(X_1,P(X,Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,P(X,Y)X_k),\end{aligned} \tag{18}$$

$$\begin{aligned}(Z(X,Y)\cdot T)(X_1,...,X_k) = &- T(Z(X,Y)X_1,X_2,...,X_k)\\ &- T(X_1,Z(X,Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,Z(X,Y)X_k),\end{aligned} \tag{19}$$

$$\begin{aligned}Q(g,T)(X_1,...,X_k;X,Y) = &- T((X\Lambda Y)X_1,X_2,...,X_k)\\ &- T(X_1,(X\Lambda Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,(X\Lambda Y)X_k),\end{aligned} \tag{20}$$

respectively [11].

By definition the *Weyl conformal curvature tensor* C is given by

$$\begin{aligned}C(X,Y)Z \;=\; & R(X,Y)Z - \frac{1}{n-2}\left[\begin{array}{c} g(Y,Z)QX - g(X,Z)QY \\ +S(Y,Z)X - S(X,Z)Y \end{array}\right]\\ &+\frac{\tau}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right],\end{aligned} \tag{21}$$

where Q denotes the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$ and τ is scalar curvature [9]. The Weyl conformal curvature tensor C is defined by $C(X,Y,Z,W)$ = $g(C(X,Y)Z,W)$. If $C = 0$, $n \geq 4$, then M is called conformally flat.

3. MAIN RESULTS

In the present section we consider the *LP*-Sasakian manifold M^n satisfying the curvature conditions $P \cdot C = 0$, $Z \cdot C = L_C Q(g,C)$, $P \cdot Z - Z \cdot P = 0$, and $P \cdot Z + Z \cdot P = 0$.

First we give the following proposition.

Proposition 1. *Let M be an n-dimensional $(n > 3)$ LP-Sasakian manifold. If the condition $P \cdot C = 0$ holds on M, then*

$$\begin{aligned}S^2(X,U) \;=\; & \Big[\frac{\tau}{n-1} - (n-1)^2 - 1\Big]S(X,U)\\ &+(n-1)[\tau - (n-1)]g(X,U)\\ &+n[\tau - n(n-1)]\eta(X)\eta(U)\end{aligned}$$

is satisfied on M, where $S^2(X,U) = S(QX,U)$.

Proof. Assume that M is an n-dimensional, $n > 3$, *LP*-Sasakian manifold satisfying the condition $P \cdot C = 0$. From (18) we have

$$\begin{aligned}(P(V,X)\cdot C)(Y,U)W = &P(V,X)C(Y,U)W \\ &- C(P(V,X)Y,U)W - C(Y,P(V,X)U)W \\ &- C(Y,U)P(V,X)W = 0, \end{aligned} \tag{22}$$

where $X, Y, U, V, W \in \chi(M)$. Taking $V = \xi$ in (22), we have

$$\begin{aligned}(P(\xi,X)\cdot C)(Y,U)W = &P(\xi,X)C(Y,U)W \\ &- C(P(\xi,X)Y,U)W - C(Y,P(\xi,X)U)W \\ &- C(Y,U)P(\xi,X)W = 0. \end{aligned} \tag{23}$$

Furthermore, substituting (2), (9), (13), (21) into (23) and multiplying with ξ, we get

$$\begin{aligned}&-g(X,C(Y,U)W) - n\eta(C(Y,U)W)\eta(X) - g(X,Y)\eta(C(\xi,U)W) \\ &+n\eta(Y)\eta(C(X,U)W) - g(X,U)\eta(C(Y,\xi)W) \\ &+n\eta(U)\eta(C(Y,X)W) + n\eta(W)\eta(C(Y,U)X) \\ &+\frac{1}{n-1}\{S(X,C(Y,U)W) + S(X,Y)\eta(C(\xi,U)W) \\ &+S(X,U)\eta(C(Y,\xi)W)\} = 0. \end{aligned} \tag{24}$$

Thus, replacing W with ξ in (24), we have

$$-g(X,C(Y,U)\xi) - n\eta(C(Y,U)X) + \frac{1}{n-1}S(X,C(Y,U)\xi) = 0. \tag{25}$$

Again, taking $Y = \xi$ in (25) and after some calculations, since $n > 3$, we get

$$\begin{aligned}S^2(X,U) = &\Big[\frac{\tau}{n-1} - (n-1)^2 - 1\Big]S(X,U) \\ &+ (n-1)[\tau - (n-1)]g(X,U) \\ &+ n[\tau - n(n-1)]\eta(X)\eta(U). \end{aligned}$$

Our theorem is thus proved. □

Theorem 2. *Let M be an n-dimensional $(n > 3)$ LP-Sasakian manifold. If the condition $Z \cdot C = L_C Q(g,C)$ holds on M, then either M is conformally flat or $L_C = \frac{\tau}{n(n-1)} - 1$.*

Proof. Let M^n be an *LP*-Sasakian manifold. So we have

$$(Z(V,X)\cdot C)(Y,U)W = L_C Q(g,C)(Y,U,W;V,X).$$

Then from (19) and (20) we can write

$$\begin{aligned} Z(V,X)C(Y,U)W - C(Z(V,X)Y,U)W - C(Y,Z(V,X)U)W \\ - C(Y,U)Z(V,X)W \\ = L_C[(V \wedge X)C(Y,U)W - C((V \wedge X)Y,U)W \\ - C(Y,(V \wedge X)U)W - C(Y,U)(V \wedge X)W]. \end{aligned} \tag{26}$$

Therefore, replacing V with ξ in (26), we have

$$\begin{aligned} Z(\xi,X)C(Y,U)W - C(Z(\xi,X)Y,U)W - C(Y,Z(\xi,X)U)W \\ - C(Y,U)Z(\xi,X)W \\ = L_C[(\xi \wedge X)C(Y,U)W - C((\xi \wedge X)Y,U)W \\ - C(Y,(\xi \wedge X)U)W - C(Y,U)(\xi \wedge X)W]. \end{aligned} \tag{27}$$

Using (20), (9) and taking the inner product of (27) with ξ, we get

$$\begin{aligned} \Big[1 - \frac{\tau}{n(n-1)} - L_C\Big][-g(X,C(Y,U)W) - \eta(C(Y,U)W)\eta(X) \\ -g(X,Y)\eta(C(\xi,U)W) + \eta(Y)\eta(C(X,U)W) \\ -g(X,U)\eta(C(Y,\xi)W) + \eta(U)\eta(C(Y,X)W) + \eta(W)\eta(C(Y,U)X)] = 0. \end{aligned} \tag{28}$$

Putting $X = Y$ in (28), we have

$$\begin{aligned} \Big[1 - \frac{\tau}{n(n-1)} - L_C\Big][-g(Y,C(Y,U)W) + \eta(W)\eta(C(Y,U)Y) \\ -g(Y,Y)\eta(C(\xi,U)W) - g(Y,U)\eta(C(Y,\xi)W)] = 0. \end{aligned} \tag{29}$$

A contraction of (29) with respect to Y gives us

$$\Big[1 - \frac{\tau}{n(n-1)} - L_C\Big]\eta(C(\xi,U)W) = 0. \tag{30}$$

If $L_C \neq 1 - \frac{\tau}{n(n-1)}$, then Eq. (30) is reduced to

$$\eta(C(\xi,U)W) = 0, \tag{31}$$

which gives

$$S(U,W) = \left(\frac{\tau}{(n-1)} - 1\right) g(U,W) + \left(\frac{\tau}{(n-1)} - n\right) \eta(U)\eta(W). \tag{32}$$

Therefore, M is a η-Einstein manifold. So, using (31) and (32), we have Eq. (28) in the form

$$C(Y,U,W,X) = 0,$$

which means that M is conformally flat.

If $L_C \neq 0$ and $\eta(C(\xi,U)W) \neq 0$, then $1 - \frac{\tau}{n(n-1)} - L_C = 0$, which gives $L_C = 1 - \frac{\tau}{n(n-1)}$. This completes the proof of the theorem. □

Corollary 3. *Every n-dimensional $(n > 3)$ nonconformally flat LP-Sasakian manifold satisfies $Z \cdot C = (1 - \frac{\tau}{n(n-1)})Q(g, C)$.*

Theorem 4. *Let M be an n-dimensional $(n > 3)$ LP-Sasakian manifold. M satisfies the condition*

$$P \cdot Z - Z \cdot P = 0$$

if and only if M is a η-Einstein manifold.

Proof. Let M satisfy the condition $P \cdot Z - Z \cdot P = 0$. Then we can write

$$\begin{aligned} & P(V,X) \cdot Z(Y,U)W - Z(V,X) \cdot P(Y,U)W \\ &= \frac{1}{n-1}[R(V,X)R_1(Y,U)W - R_1(V,X)R(Y,U)W] \\ &\quad + \frac{\tau}{n(n-1)^2}[R_1(V,X)R_0(Y,U)W - R_0(V,X)R_1(Y,U)W] \\ &\quad + \frac{\tau}{n(n-1)}[R_0(V,X)R(Y,U)W - R(V,X)R_0(Y,U)W] = 0. \end{aligned} \tag{33}$$

Therefore, replacing V with ξ in (33), we have

$$\begin{aligned} & P(\xi,X) \cdot Z(Y,U)W - Z(\xi,X) \cdot P(Y,U)W \\ &= \frac{1}{n-1}[R(\xi,X)R_1(Y,U)W - R_1(\xi,X)R(Y,U)W] \\ &\quad + \frac{\tau}{n(n-1)^2}[R_1(\xi,X)R_0(Y,U)W - R_0(\xi,X)R_1(Y,U)W] \\ &\quad + \frac{\tau}{n(n-1)}[R_0(\xi,X)R(Y,U)W - R(\xi,X)R_0(Y,U)W] = 0. \end{aligned} \tag{34}$$

Using (10), (13), we get

$$\begin{aligned} & \frac{1}{n-1}[S(U,W)g(X,Y)\xi - S(U,W)\eta(Y)X - g(X,U)S(Y,W)\xi \\ &\quad + S(Y,W)\eta(U)X - S(X,R(Y,U)W)\xi + (n-1)g(U,W)\eta(Y)X \\ &\quad - (n-1)g(Y,W)\eta(U)X] \\ &\quad + \frac{\tau}{n(n-1)^2}[g(U,W)g(X,Y)\xi - g(U,W)\eta(Y)X - g(Y,W)g(X,U)\xi \\ &\quad + g(Y,W)\eta(U)X - S(U,W)g(X,Y)\xi + S(U,W)\eta(Y)X \\ &\quad + S(Y,W)g(X,U)\xi - S(Y,W)\eta(U)X] \\ &\quad + \frac{\tau}{n(n-1)}[g(X,R(Y,U)W)\xi + g(Y,W)\eta(U)X - g(U,W)g(X,Y)\xi \\ &\quad + g(Y,W)g(X,U)\xi - g(Y,W)\eta(U)X] = 0. \end{aligned} \tag{35}$$

Again, taking $U = \xi$ in (35), we get

$$\begin{aligned}
&\frac{1}{n-1}[(n-1)g(X,Y)\eta(W)\xi - S(Y,W)\eta(X)\xi - S(Y,W)X \\
&\quad +(n-1)g(Y,W)\eta(X)\xi - S(X,Y)\eta(W)\xi + (n-1)g(Y,W)X] \\
&\quad +\frac{\tau}{n(n-1)^2}[g(X,Y)\eta(W)\xi - \eta(W)\eta(Y)X - g(Y,W)\eta(X)\xi - g(Y,W)X \\
&\quad -(n-1)g(X,Y)\eta(W)\xi + (n-1)\eta(W)\eta(Y)X \\
&\quad -S(Y,W)\eta(X)\xi + S(Y,W)X] = 0. \qquad (36)
\end{aligned}$$

Taking the inner product of (36) with ξ, we find

$$\begin{aligned}
&\frac{1}{n-1}[S(X,Y)\eta(W) - (n-1)g(X,Y)\eta(W)] \\
&\qquad +\frac{\tau(n-2)}{n(n-1)^2}[g(X,Y)\eta(W) + \eta(X)\eta(Y)\eta(W)] = 0. \qquad (37)
\end{aligned}$$

Again, taking $W = \xi$ and using (3) in (37), we get

$$\begin{aligned}
S(X,Y) = &\left[(n-1) - \frac{(n-2)}{n(n-1)}\tau\right] g(X,Y) \\
&- \left[\frac{(n-2)}{n(n-1)}\tau\right] \eta(X)\eta(Y). \qquad (38)
\end{aligned}$$

So, M is a η-Einstein manifold.

Conversely, if M^n is a η-Einstein manifold, then it is easy to show that $P \cdot Z - Z \cdot P = 0$. Our theorem is thus proved. □

Theorem 5. *Let M be an n-dimensional $(n > 3)$ LP-Sasakian manifold. M satisfies the condition*

$$P \cdot Z + Z \cdot P = 0$$

if and only if M is an Einstein manifold.

Proof. Let M satisfy the condition $P \cdot Z + Z \cdot P = 0$. Then, from (33) and (34), we can write

$$\begin{aligned}
&2R(\xi,X)R(Y,U)W \\
&\quad -\frac{1}{n-1}[R(\xi,X)R_1(Y,U)W + R_1(\xi,X)R(Y,U)W] \\
&\quad +\frac{\tau}{n(n-1)^2}[R_1(\xi,X)R_0(Y,U)W + R_0(\xi,X)R_1(Y,U)W] \\
&\quad -\frac{\tau}{n(n-1)}[R_0(\xi,X)R(Y,U)W + R(\xi,X)R_0(Y,U)W] = 0. \qquad (39)
\end{aligned}$$

Using (6), (10), and (13) in (39), we have

$$
\begin{aligned}
&2[g(X,R(Y,U)W)\xi - g(U,W)\eta(Y)X + g(Y,W)\eta(U)X]\\
&\quad -\frac{1}{n-1}[S(U,W)g(X,Y)\xi - S(U,W)\eta(Y)X - S(Y,W)g(X,U)\xi\\
&\quad +S(Y,W)\eta(U)X + S(X,R(Y,U)W)\xi - (n-1)g(U,W)\eta(Y)X\\
&\quad +(n-1)g(Y,W)\eta(U)X]\\
&\quad +\frac{\tau}{n(n-1)^2}[g(U,W)S(X,Y)\xi - (n-1)g(U,W)\eta(Y)X\\
&\quad -g(Y,W)S(X,U)\xi + (n-1)g(Y,W)\eta(U)X + S(U,W)g(X,Y)\xi\\
&\quad -S(U,W)\eta(Y)X - S(Y,W)g(X,U)\xi + S(Y,W)\eta(U)X]\\
&\quad -\frac{\tau}{n(n-1)}[g(X,R(Y,U)W)\xi - 2g(U,W)\eta(Y)X + 2g(Y,W)\eta(U)X\\
&\quad +g(U,W)g(X,Y)\xi - g(Y,W)g(X,U)\xi] = 0. \qquad (40)
\end{aligned}
$$

Replacing Y with ξ and using (3) in (40), we have

$$
\begin{aligned}
&2[g(X,R(\xi,U)W)\xi + g(U,W)X + \eta(W)\eta(U)X]\\
&\quad -\frac{1}{n-1}[S(U,W)\eta(X)\xi + S(U,W)X - (n-1)g(X,U)\eta(W)\xi\\
&\quad +2(n-1)\eta(W)\eta(U)X + S(X,R(\xi,U)W)\xi + (n-1)g(U,W)X]\\
&\quad +\frac{\tau}{n(n-1)^2}[(n-1)g(U,W)\eta(X)\xi + (n-1)g(U,W)X\\
&\quad -S(X,U)\eta(W)\xi + (n-1)\eta(W)\eta(U)X + S(U,W)\eta(X)\xi\\
&\quad +S(U,W)X - (n-1)g(X,U)\eta(W)\xi + (n-1)\eta(W)\eta(U)X]\\
&\quad -\frac{\tau}{n(n-1)}[g(X,R(\xi,U)W)\xi + 2g(U,W)X + 2\eta(W)\eta(U)X\\
&\quad +g(U,W)\eta(X)\xi - g(X,U)\eta(W)\xi] = 0. \qquad (41)
\end{aligned}
$$

Taking the inner product of (41) with ξ and using (7), (10), we get

$$
\begin{aligned}
&\left[2 - \frac{2\tau}{n(n-1)}\right][g(X,U)\eta(W) + \eta(X)\eta(U)\eta(W)]\\
&\quad + \left[\frac{\tau}{n(n-1)^2} - \frac{1}{n-1}\right][(n-1)g(X,U)\eta(W) + 2(n-1)\eta(X)\eta(U)\eta(W)\\
&\quad +S(X,U)\eta(W)] = 0. \qquad (42)
\end{aligned}
$$

Again, taking $W = \xi$ and using (3) in (42), we get

$$
\begin{aligned}
&\left[\frac{2\tau}{n(n-1)} - 2\right][g(X,U) + \eta(X)\eta(U)]\\
&\quad - \left[\frac{\tau}{n(n-1)^2} - \frac{1}{n-1}\right][(n-1)g(X,U)\\
&\quad +2(n-1)\eta(X)\eta(U) + S(X,U)] = 0. \qquad (43)
\end{aligned}
$$

Thus, from (43), we have

$$S(X,U) = (n-1)g(X,U).$$

So, M^n is an Einstein manifold.

Conversely, if M^n is an Einstein manifold, then it is easy to show that $P \cdot Z + Z \cdot P = 0$. Our theorem is thus proved. $\square$

ACKNOWLEDGEMENT

This study was supported by the Dumlupınar University research foundation (project No. 2004-9).

REFERENCES

1. Matsumoto, K. On Lorentzian paracontact manifolds. *Bull. of Yamagata Univ. Nat. Sci.*, 1989, **12**, 151–156.
2. Mihai, I. and Rosca, R. *On Lorentzian P-Sasakian Manifolds, Classical Analysis*. World Scientific, Singapore, 1992, 155–169.
3. Matsumoto, K. and Mihai, I. On a certain transformation in a Lorentzian para-Sasakian manifold. *Tensor, N. S.*, 1988, **47**, 189–197.
4. Tripathi, M. M. and De, U. C. Lorentzian almost paracontact manifolds and their submanifolds. *J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.*, 2001, **8**, 101–105.
5. Blair, D. E., Kim, J. S. and Tripathi, M. M. On the concircular curvature tensor of a contact metric manifold. *J. Korean Math. Soc.*, 2005, **42**, 883–892.
6. Tigaeru, C. *v*-projective symmetries of fibered manifolds. *Arch. Math.*, 1998, **34**, 347–352.
7. Sato, I. On a structure similar to almost contact structures. *Tensor, N. S.*, 1976, **30**, 219–224.
8. Sato, I. On a structure similar to almost contact structures II. *Tensor, N. S.*, 1977, **31**, 199–205.
9. Yano, K. and Kon, M. *Structures on Manifolds*. Series in Pure Mathematics, Vol. 3, 1984. World Scientific, Singapore.
10. Blair, D. E. *Contact Manifolds in Riemannian Geometry.* Lecture Notes in Mathematics, Vol. 509, 1976, Springer-Verlag, Berlin.
11. Deszcz, R. On pseudosymmetric spaces. *Bull. Soc. Math. Belg.*, 1990, **49**, 134–145.

Ühest Lorentzi para-Sasaki muutkondade klassist

Cengizhan Murathan, Ahmet Yıldız, Kadri Arslan ja Uday Chand De

On käsitletud Lorentzi para-Sasaki muutkondi, mille puhul $P \cdot C = 0$, $Z \cdot C = L_C Q(g,C)$, $P \cdot Z - Z \cdot P = 0$ või $P \cdot Z + Z \cdot P = 0$, kus C on Weyli konformse kõveruse tensor, P on $v-$Weyli projektiivne tensor ja Z on kontsirkulaartensor.