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The influence of delamination on free vibrations of composite beams on Pasternak soil

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Abstract. Free vibrations of delaminated beams resting on Pasternak soil are analysed. Differential stretching and bending–extension coupling are considered in the formulation. The influence of soil parameters, size and location of the delamination on the frequencies and mode shapes is investigated. Some numerical examples and comparisons are presented.

Key words: delamination, vibrations, beams, elastic foundation.

1. INTRODUCTION

Dynamic response of composite laminates has received a great deal of attention. Composites are very sensitive to the defects and anomalies induced during their fabrication and exploitation. One of the most common defects in composite laminates is delamination. The presence of delamination may cause changes in the vibration characteristics of the structure and can be the most damaging failure mode of composite materials. A detailed review on this topic was recently given by Zou et al. [1], and Luo and Hanagud [2]. Free vibrations of delaminated composite beams were first studied by Ramkumar et al. [3]. They proposed to deal with a laminated beam using four Timoshenko beams connected at the delamination edges. Wang et al. [4] examined free vibrations of delaminated beams including the effect of coupling between flexural and longitudinal motion. The intact and delaminated parts have been treated as classical beam models. In this model the delaminated layers are assumed to deform "freely". Mujumdar and Suryanarayan [5] showed that this assumption is physically inadmissible because of possible overlapping of delaminated layers for off-midplane delaminations. They suggested a model where delaminated layers are "constrained", having identical transverse deformations. Tracy and

Pardoen [6] presented a simplified analytical model to predict stiffness degradation in composite laminates without bending/extensional coupling. The constrained models fail to explain the delamination opening modes found in experiments [7]. In order to take into account constraints between upper and lower delamination parts, Luo and Hanagud [2] proposed the piecewise linear spring model to simulate the nonlinear interaction between delaminated surfaces. The nonlinear constraint model for describing the behaviour of delaminated parts was introduced by Wang and Tong [8]. Brandinelli and Massabo [9] proposed a linear elastic bridging-mechanism acting along the surfaces of the delamination. This model is based on the first-order shear deformation theory of laminated plates.

In most papers the vibrations of beams have been studied in the case where the beam has a single delamination or multiple delaminations of the same length through the beam thickness. A sandwich beam with double delaminations is investigated in [10]. Free vibrations of beams with two enveloping delaminations were studied by Shu and Della [11]. Multi-delaminated composite beams subjected to the axial compression load are considered in [12]. Nonlinear vibration of composite beams with arbitrary delamination is discussed in [13]. The layerwise theory of composite laminates is explored in [12,14].

Many practical problems related to soil–structure interactions can be modelled by means of a beam on elastic foundation. Various types of foundation models such as Winkler, Pasternak, Vlasov, etc. models have been used in the analysis of beams on elastic foundation [15]. In the case of the Winkler model it is assumed that the foundation applies only to a reaction proportional to the beam deflection. The medium is taken into account as a system composed of infinitely close linear springs. The interaction between springs is not considered in the Winkler model. Pasternak [16] has proposed a physically close and mathematically simple two-parameter foundation model with shear interactions. The first foundation parameter is the same as in the Winkler model and the second one is the stiffness of the shearing layer in the Pasternak foundation model. The vibrations of solid beams resting on elastic foundation have been investigated in a number of papers [17–24].

It appears that there is no analytical model for delaminated beams resting on elastic foundation. In the present paper the ideas of [4,5,11,21] are extended to the case of delaminated beams resting on Pasternak soil. Two different models are applied. In the first model the coupling between longitudinal vibration and transverse bending is taken into account, whereas in the second model the delaminated parts are constrained having identical transverse displacements. The shear deformation and rotational inertia terms are not included in the present study.

2. FORMULATION OF THE PROBLEM

Consider a beam of length L_0 resting on Pasternak soil and having an arbitrary delamination of length L_2 (Fig. 1). The through-width delamination is assumed to be parallel to the beam axis and to be located arbitrarily in both the spanwise and thicknesswise directions. The delaminated section is divided into two sublaminates of thickness h_2 and h_3, respectively. Thus, the beam is divided into four sections denoted I–IV as shown in Fig. 1. Each section is treated as a classical Bernoulli beam model, provided that $L_i >> h$. For the sake of simplicity only one delamination zone is considered. The beam with multiple delaminations can be treated analogously.

Governing equations for the vibrations of beam sections on the two-parameter elastic foundation can be deduced by means of Hamilton's principle [25]:

$$\frac{\partial^2}{\partial x^2}\left[D_i(x)\frac{\partial^2 w_i}{\partial x^2}\right]+k_w(x)w_i-\frac{\partial}{\partial x}\left[k_p(x)\frac{\partial w_i}{\partial x}\right]=-\rho_i(x)A_i(x)\frac{\partial^2 w_i}{\partial t^2}, \tag{1}$$

where $w_i(x)$ is the vertical displacement of the ith beam section, $D_i(x)$ is the bending stiffness, $\rho_i(x)$ is the material density, x is the axial coordinate, and t is the time. The quantity A_i denotes the cross-sectional area, whereas $k_w(x)$ and $k_p(x)$ denote the variable coefficients of the Winkler and Pasternak foundations, respectively. The bending stiffness for homogeneous and isotropic beams is given as

$$D_i(x)=E_i(x)I_i(x). \tag{2}$$

The Winkler foundation parameter, denoted by k_w, represents the distributed translational resistance of the foundation, and the Pasternak parameter k_p accounts for the rotational stiffness [22]. In the case of a plane stress problem, E is Young's modulus, whereas in the case of the plane strain problem, E would be replaced by an equivalent Young's modulus $E=E/(1-\nu^2)$, where ν is Poisson's ratio. The quantity I denotes the second moment of the cross-sectional area of the beam segment. In the case of composite laminates, the quantity D can be calculated using the classical laminate theory [26] in the following way:

$$D=D_{11}-\frac{B_{11}^2}{A_{11}}, \tag{3}$$

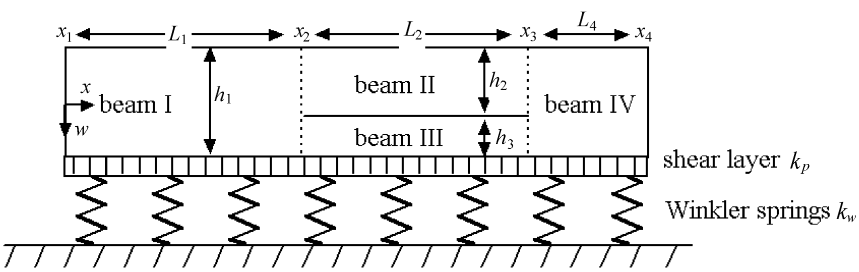

Fig. 1. A model of a beam on elastic foundation.

where

$$
\begin{aligned}
D_{11} &= \frac{b}{3}\sum_{k=1}^{n}\hat{Q}_{11}^{k}(z_k^3 - z_{k-1}^3),\\
A_{11} &= b\sum_{k=1}^{n}\hat{Q}_{11}^{k}(z_k - z_{k-1}),\\
B_{11} &= \frac{b}{2}\sum_{k=1}^{n}\hat{Q}_{11}^{k}(z_k^2 - z_{k-1}^2)
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\hat{Q}_{11}^{k} = Q_{11}^{k}\cos^4\theta + Q_{22}^{k}\sin^4\theta + 2\cos^2\theta\sin^2\theta(Q_{11}^{k} + 2Q_{66}^{k}),\\
&Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \qquad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}},\\
&Q_{66} = G_{12}, \qquad \nu_{21} = \frac{\nu_{12}E_{22}}{E_{11}}.
\end{aligned}
\tag{5}
$$

In Eqs (3)–(5) D_{11} denotes the bending stiffness, A_{11} is the extensional stiffness, and B_{11} is the coupling bending/extensional stiffness of the beam section. The quantity $\hat{Q}_{11}^{k}$ is the coefficient stiffness of the kth lamina of the beam section and can be calculated with the aid of lamina parameters. In Eqs (5) E_{11}, E_{22}, and ν_{12}, ν_{21} stand for longitudinal and transverse Young's moduli and Poisson's ratio of a single lamina, respectively; θ is the lamina orientation angle; z_k and z_{k-1} are the coordinates of the kth lamina with respect to the mid-plane of the beam section.

For free vibrations the solution can be sought in the form

$$
w_i(x,t) = W_i(x)e^{j\omega t}, \tag{6}
$$

where ω is the natural frequency and $W(x)$ is the mode shape of the ith beam section and $j = \sqrt{-1}$. Equation (1) then yields

$$
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[D_i(x)\frac{\mathrm{d}^2W_i}{\mathrm{d}x^2}\right] + k_w(x)W_i - \frac{\mathrm{d}}{\mathrm{d}x_i}\left[k_p(x)\frac{\mathrm{d}W_i}{\mathrm{d}x}\right] - \rho_i(x)A_i\omega^2W_i = 0. \tag{7}
$$

The solution of (7) for the general case of $k_w(x)$ and $k_p(x)$ is not available. In the further study it is assumed that $k_w = \mathrm{const}$, $k_p = \mathrm{const}$, and $\rho_i = \mathrm{const}$.

It is convenient to rewrite this equation in the form

$$
\frac{\mathrm{d}^4W_i}{\mathrm{d}x^4} - 4\delta_i\frac{\mathrm{d}^2W_i}{\mathrm{d}x^2} + 4\lambda_iW_i = 0, \tag{8}
$$

where

$$\delta_i = \frac{k_p}{4D_i}, \quad \lambda_i = \frac{k_w - \rho_i A_i \omega^2}{4D_i}. \tag{9}$$

The solution of (8) can be expressed as

$$W_i = \sum_{k=1}^{4} C_k^i F_k^i(x), \tag{10}$$

where the functions $F_k^i(x)$ can be presented as follows:

$$\begin{Bmatrix} F_1^i(x) \\ F_2^i(x) \\ F_3^i(x) \\ F_4^i(x) \end{Bmatrix} = \begin{Bmatrix} \cosh \gamma_i x \\ \sinh \gamma_i x \\ \cosh \nu_i x \\ \sinh \nu_i x \end{Bmatrix}, \text{ if } \delta_i^2 > \lambda_i \text{ and } \lambda_i \le 0, \tag{11}$$

$$\begin{Bmatrix} F_1^i(x) \\ F_2^i(x) \\ F_3^i(x) \\ F_4^i(x) \end{Bmatrix} = \begin{Bmatrix} \cosh \gamma_i x \\ \sinh \gamma_i x \\ \cos \mu_i x \\ \sin \mu_i x \end{Bmatrix}, \text{ if } \delta_i^2 > \lambda_i \text{ and } \lambda_i > 0, \tag{12}$$

$$\begin{Bmatrix} F_1^i(x) \\ F_2^i(x) \\ F_3^i(x) \\ F_4^i(x) \end{Bmatrix} = \begin{Bmatrix} \cos \beta_i x \cosh \alpha_i x \\ \cos \beta_i x \sinh \alpha_i x \\ \sin \beta_i x \cosh \alpha_i x \\ \sin \beta_i x \sinh \alpha_i x \end{Bmatrix}, \text{ if } \delta_i^2 < \lambda_i. \tag{13}$$

It can be stated that for the most real physical parameters the mathematically exact statement $\delta_i^2 = \lambda_i$ is not achieved. The quantities α_i, β_i, γ_i, μ_i, ν_i are defined as

$$\begin{gathered} \alpha_i = \sqrt{\sqrt{\lambda_i} + \delta_i}, \qquad \beta_i = \sqrt{\sqrt{\lambda_i} - \delta_i}, \qquad \gamma_i = \sqrt{2}\sqrt{\delta_i + \sqrt{\delta_i^2 - \lambda_i}}, \\ \mu_i = \sqrt{2}\sqrt{\sqrt{\delta_i^2 - \lambda_i} - \delta_i}, \qquad \nu_i = \sqrt{2}\sqrt{\delta_i - \sqrt{\delta_i^2 - \lambda_i}}. \end{gathered} \tag{14}$$

For each beam section the governing equations are solved, whereas the unknown integration constants can be found with the aid of boundary and continuity conditions between beam sections. Since all equations contain the frequency ω, a solution can be obtained. The type of the solution depends on the beam geometry, elastic foundation, and delamination size and position.

3. FREE MODEL

In the following analysis the nondimensional quantities

$$W_i^* = \frac{W_i}{L_i}, \quad i = 1, ..., 4 \tag{15}$$

will be introduced.

In the split region, in addition to the deflection, the governing equations for the longitudinal equilibrium are written as follows [4]:

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} + \frac{\omega^2}{c_i^2} u_i = 0, \quad i = 2, 3, \tag{16}$$

where $c_i^2 = E_i / \rho_i$ and u_i denote the longitudinal displacement of the ith beam section. The general solution of (16) may be written in the following form:

$$u_i = G_i \sin\left(\frac{\omega}{c_i} x\right) + H_i \cos\left(\frac{\omega}{c_i} x\right). \tag{17}$$

The 20 coefficients C_k^i, G_m, and H_m $(i, k = 1, \ldots, 4;\ m = 2, 3)$ are determined with the aid of the boundary and continuity conditions. If the beam is clamped at $x = x_1$ (Fig. 1), then $W_1 = 0$ and $W_1' = 0$; if simply supported, then $W_1 = 0$ and $W_1'' = 0$; if free, then $W_1'' = 0$ and $W_1''' = 0$. The analogous boundary conditions can be established at $x = x_4$. The continuity conditions for deflection, slope, shear force, bending moment, and longitudinal displacement at $x = x_2$ are:

$$\begin{aligned}
&W_1^* = W_2^*, \qquad W_1^* = W_3^*, \\
&W_1^{*\prime} = W_2^{*\prime}, \qquad W_1^{*\prime} = W_3^{*\prime}, \\
&D_1 W_1^{*\prime\prime\prime} = D_2 W_2^{*\prime\prime\prime} + D_3 W_3^{*\prime\prime\prime}, \\
&D_1 W_1^{*\prime\prime} = D_2 W_2^{*\prime\prime} + D_3 W_3^{*\prime\prime} + e E_3 A_3 u_3', \\
&E_2 A_2 u_2' = E_3 A_3 u_3', \\
&u_2 = u_3 - e W_2^{*\prime},
\end{aligned} \tag{18}$$

where e is the distance between neutral planes of the upper and lower parts of the split region. The quantities A_2 and A_3 denote the cross-sectional areas of delaminated beam sections. Similar continuity conditions can be written at $x = x_3$. A nontrivial solution for the coefficients exists only when the determinant of the coefficient matrix vanishes. The frequencies and mode shapes can be calculated as eigenvalue and eigenvector solutions, respectively.

4. CONSTRAINED MODEL

In order to preclude the nonphysical vibration modes, the deflections of segments II and III can be forced to vibrate together $(w_2 = w_3)$. In this model the governing equations are

$$D_i \frac{\partial^4 w_i}{\partial x^4} - k_p \frac{\partial^2 w_i}{\partial x^2} + k_w w_i + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \ \ (i = 1 \text{ and } i = 4). \tag{19}$$

For the second and third parts we have

$$(D_2 + D_3) \frac{\partial^4 w_2}{\partial x^4} - k_p \frac{\partial^2 w_2}{\partial x^2} + k_w w_2 + (\rho_2 A_2 + \rho_3 A_3) \frac{\partial^2 w_2}{\partial t^2} = 0. \tag{20}$$

The generalized solution for W_i^* $(i = 1, 2, 4)$ in the case of the "constrained model" is identical to (10)–(13). The number of unknown constants C_k^i is now 12, which can be determined by four boundary conditions and eight continuity conditions (four at $x = x_2$ and four at $x = x_3$). The continuity conditions at $x = x_2$ for deflections, slopes, and shear forces can be written as

$$\begin{aligned} &W_1^* = W_2^*, \qquad W_1^{*\prime} = W_2^{*\prime}, \\ &D_1 W_1^{*\prime\prime\prime} = (D_2 + D_3) W_2^{*\prime\prime\prime}. \end{aligned} \tag{21}$$

The continuity condition for bending moments can be presented as

$$M_1 = M_2 + M_3 - \frac{1}{2} P_2 (h_1 - h_2) + \frac{1}{2} P_3 (h_1 - h_3), \tag{22}$$

where $M_i = -D_i W_i^{*\prime\prime}$. The axial forces P_2 and P_3 can be established from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium [5]:

$$\begin{aligned} &\frac{P_3 L_2}{E_3 A_3} - \frac{P_2 L_2}{E_2 A_2} = \frac{h_1}{2} (W_1^{*\prime}(x_2) - W_4^{*\prime}(x_3)), \\ &P_2 + P_3 = 0. \end{aligned} \tag{23}$$

Similarly, we can derive the continuity conditions at $x = x_3$.

5. NUMERICAL RESULTS

It is convenient to define the following nondimensional quantities:

$$\Omega^2 = \frac{\rho_1 A_1 L_0^4}{E_1 I_1} \omega^2, \quad k_w^* = \frac{k_w L_0^4}{E_1 I_1}, \quad k_p^* = \frac{k_p L_0^4}{E_1 I_1}. \tag{24}$$

The following cases are considered:

1. Homogeneous beam without delamination on elastic foundation

A comparison with the results given in [24,25] is shown in Table 1, where the first three nondimensional natural frequencies (Ω) of a homogeneous clamped–clamped beam are given as functions of the two soil parameters. It should be noted that $\bar{k}_p = k_p^* \pi^2$, $h_1/L_0 = 1/120$ and the frequency parameter $\sqrt{\Omega}$ is used in [24,25].

2. Clamped–clamped delaminated beam without foundation

Comparisons with the results given in [4,11,14] are presented in Table 2, where the first three nondimensional natural frequencies (Ω) are given as functions of the delamination length. The single symmetric midplane delamination is applied here. Here $k_1 = k_2 = 0$ and $h_1/L_0 = 1/120$.

Tables 1 and 2 show good agreement between the present model and the analytical [4,25], numerical [11,24], and FEM [14] results.

3. Delaminated composite beam without foundation

The third verification is performed on a T300/934 graphite/epoxy cantilever beam with a $[0°/90°]_{2s}$ stacking sequence, which was studied in [2,7,11]. The dimensions of the 8-ply beam are $127 \times 12.7 \times 1.016\ \text{mm}^3$. The material properties for the lamina are: $E_{11} = 134\ \text{GPa}$, $E_{22} = 10.3\ \text{GPa}$, $G_{12} = 5\ \text{GPa}$, $\nu_{12} = 0.33$, and $\rho = 1.4 \times 10^3\ \text{kg/m}^3$. The delaminations are either at the midspan or at $h_2/h_1 = 0.25$ and the lengths are 25.4, 50.8, 76.2, and 101.6 mm. The primary frequencies are shown in Table 3. The first row corresponds to the midplane delamination and the second row corresponds to the delamination at the position $h_2/h_1 = 0.25$. Good agreement was obtained between the frequencies predicted by the present method and the experimental and analytical results by Shen and Grady [7] and analytical results by Shu and Della [11], and Luo and Hanagud [2]. The values of delamination length strongly influence the values of frequencies.

Table 1. The first three natural frequencies of the beam on elastic foundation

Foundation parameter k_w^*	Foundation parameter $\bar{k}_p$	Present	Chen et al. [24]	De Rosa and Maurizi [25]
0.0	0.0	22.3733	22.3861	22.3729
		61.6649	61.6743	61.6853
		120.9032	120.7977	120.9120
0.0	1.0	25.0887	24.9380	24.9400
		65.2412	65.2492	65.2541
		125.0640	124.7689	124.8583
100.0	0.0	24.6090	24.5174	24.5025
		62.4716	62.4795	62.4732
		121.3324	123.1212	121.0140
100.0	1.0	27.0997	26.8676	26.8531
		66.0026	66.0108	65.9994
		125.4781	125.1669	125.2609

Table 2. The first three natural frequencies of the beam with a midplane delamination

Delamination length L_2/L_0	Present	Shu and Della [11]	Wang et al. [4]	Lee [14]
0.0	22.37	22.37	22.39	22.36
	61.67	61.67	61.67	61.61
	120.90	120.90	121.91	120.68
0.1	22.37	22.37	22.37	22.36
	60.81	60.81	60.76	60.74
	120.83	120.83	120.81	120.62
0.2	22.35	22.36	22.35	22.35
	56.00	56.00	55.97	55.95
	118.87	118.87	118.76	118.69
0.3	22.24	22.24	22.23	22.23
	49.00	49.00	49.00	48.97
	109.16	109.16	109.04	109.03
0.4	21.83	21.83	21.83	21.82
	43.89	43.89	43.87	43.86
	93.59	93.59	93.57	93.51
0.5	20.89	20.89	20.88	20.88
	41.52	41.52	41.45	41.50
	82.30	82.29	82.29	82.23
0.6	19.29	19.30	19.29	19.28
	41.04	41.04	40.93	41.01
	77.69	77.69	77.64	77.64
0.7	17.23	17.23	17.23	17.22
	40.82	40.82	40.72	40.80
	77.18	77.18	77.05	77.12
0.8	15.05	15.05	15.05	15.05
	39.06	39.07	39.01	39.04
	75.43	75.43	75.33	75.39
0.9	13.00	13.00	13.00	12.99
	35.39	35.39	35.38	35.38
	69.19	69.19	69.17	69.16

Table 3. Primary frequencies for a symmetric model without foundation

Delamination length, mm	Present, Hz	Reference [11], Hz	Reference [7], Hz	Reference [2], Hz
0.0	82.02	81.88	82.04	81.86
	82.02	81.88	82.04	81.86
25.4	80.92	80.47	80.13	81.84
	81.86	81.53	81.46	82.02
50.8	76.03	75.36	75.29	76.81
	80.66	80.13	79.93	80.79
76.2	66.84	66.14	66.94	67.64
	77.72	77.03	76.71	77.82
101.6	56.26	55.67	57.24	56.95
	73.07	72.28	71.66	73.15

4. Homogeneous delaminated clamped–clamped beam on Pasternak soil

Table 4 tabulates the first two natural frequencies (Ω_1, Ω_2) as functions of foundation coefficients and delamination length. The beam consists of one symmetric midplane delamination zone of variable length. The first and second rows correspond to the first and second natural frequencies, respectively.

The influence of foundation parameters and delamination length on the first and second mode shapes is presented in Figs 2–5. In the case of symmetric delamination the differences between computed frequencies of the free and constrained models are insignificant. These differences become important in the case of long nonsymmetric delaminations.

Variation of the primary and secondary frequencies due to the delamination thicknesswise locations is presented in Table 5. Note that the frequencies of both models are almost equal when the delamination is near midplane, whereas the frequencies calculated by applying free model are smaller when the delamination location shifts towards the beam surface. This fact can be explained by overlapping of the delaminated layers in free model [5].

Table 4. The first two natural frequencies for the clamped–clamped delaminated beam

Delamination length L_2/L_0	$k_w^* = 0$, $\bar{k}_p = 1.0$	$k_w^* = 100.0$, $\bar{k}_p = 0.0$	$k_w^* = 100.0$, $\bar{k}_p = 1.0$
0.0	25.09	24.61	27.10
	65.24	62.47	66.00
0.1	25.70	25.01	28.01
	64.67	61.64	65.45
0.2	26.37	25.45	29.00
	61.68	57.00	62.58
0.3	26.83	25.73	29.75
	57.41	50.37	58.55
0.4	26.97	25.68	30.14
	54.10	45.67	55.51
0.5	26.63	25.09	30.02
	52.44	43.60	54.07
0.6	25.76	23.88	29.34
	52.06	43.29	53.83
0.7	24.42	22.28	28.21
	51.95	43.17	53.81
0.8	22.75	20.65	26.78
	50.69	41.54	52.62
0.9	20.89	19.21	25.23
	47.52	38.11	49.58

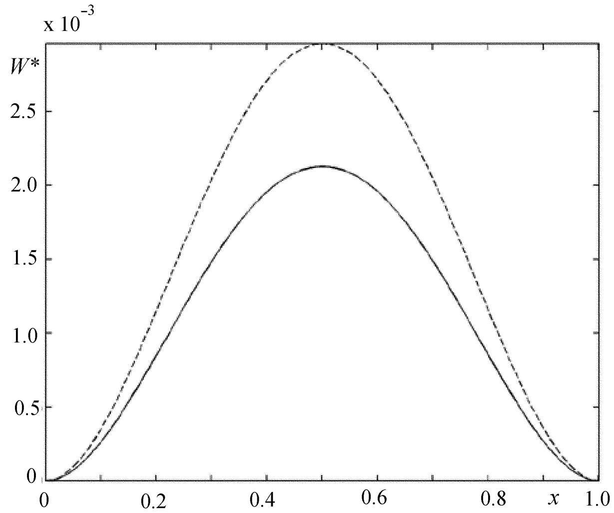

Fig. 2. The first mode shape for a beam without delamination: solid line $k_w^* = 100.0$, $\bar{k}_p = 1.0$; dashed line $k_w^* = \bar{k}_p = 0$.

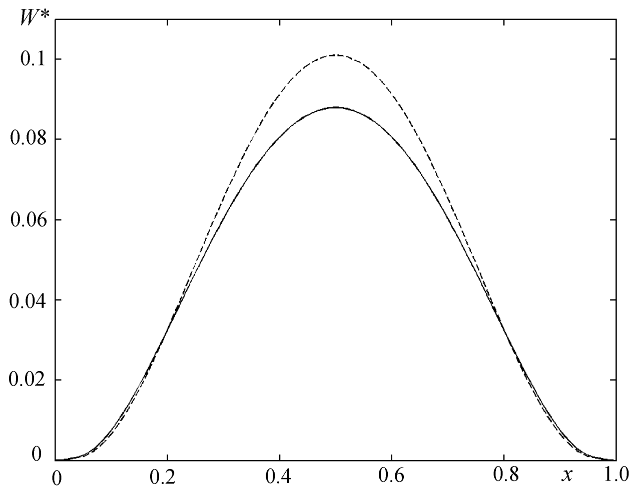

Fig. 3. The first mode shape for a beam with delamination $L_2/L_0 = 0.9$: solid line $k_w^* = 100.0$, $\bar{k}_p = 1.0$; dashed line $k_w^* = \bar{k}_p = 0$.

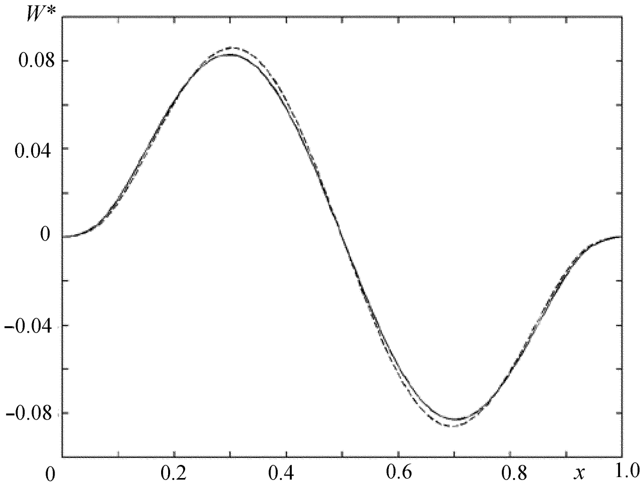

Fig. 4. The second mode shape for a beam without delamination: solid line $k_w^* = 100.0$, $\bar{k}_p = 1.0$; dashed line $k_w^* = \bar{k}_p = 0$.

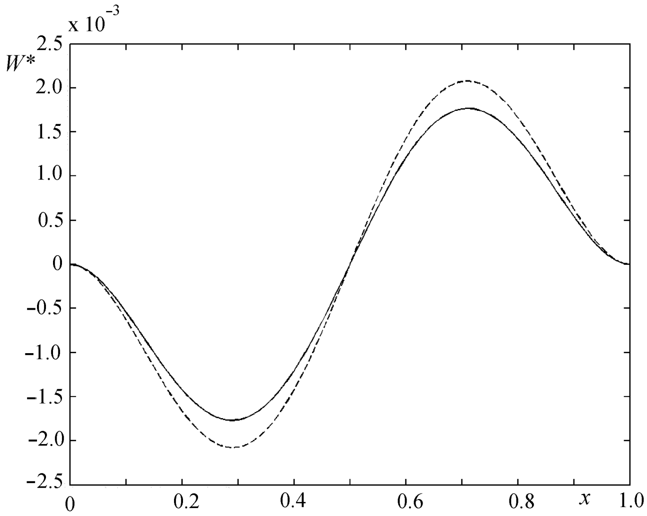

Fig. 5. The second mode shape for a beam with delamination $L_2/L_0 = 0.9$: solid line $k_w^* = 100.0$, $\bar{k}_p = 1.0$; dashed line $k_w^* = \bar{k}_p = 0$.

In Fig. 6 the two nondimensional frequencies are given as function of the first foundation parameter for $\bar{k}_p = 2.0$. In Fig. 7 the two nondimensional frequencies are given as function of the second foundation parameter for $k_w^* = 100.0$. In both cases the beam has the symmetric midplane delamination $L_2/L_0 = 0.6$.

Table 5. Nondimensional first frequencies of clamped–clamped beams with nonsymmetric delamination: $k_w^* = 4$, $\bar{k}_p = 0$, $L_2/L_0 = 0.8$, $L_1/L_0 = 0.1$

h_2/h_1	Ω_1		Ω_2	
	Free model	Constrained model	Free model	Constrained model
0.50	15.14	15.10	39.10	39.08
0.40	13.40	15.73	35.64	40.66
0.30	10.49	17.29	28.21	44.66
0.20	7.42	19.16	19.33	49.90
0.15	6.03	20.07	14.73	52.74

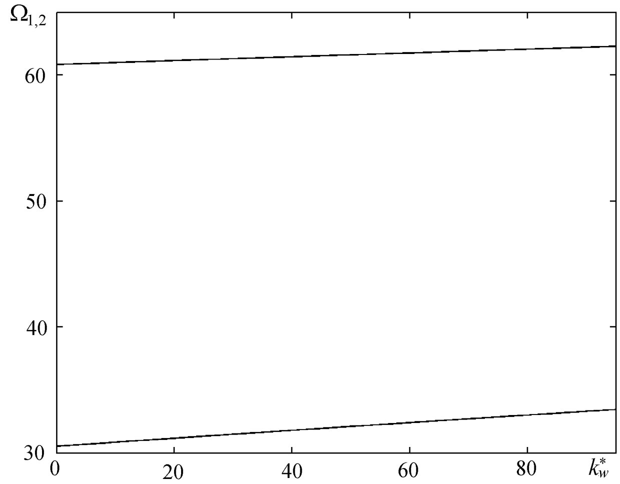

Fig. 6. The lowest two natural frequencies of a clamped-clamped beam versus the first foundation parameter; $L_2/L_0 = 0.6$, $h_1/L_0 = 1/120$.

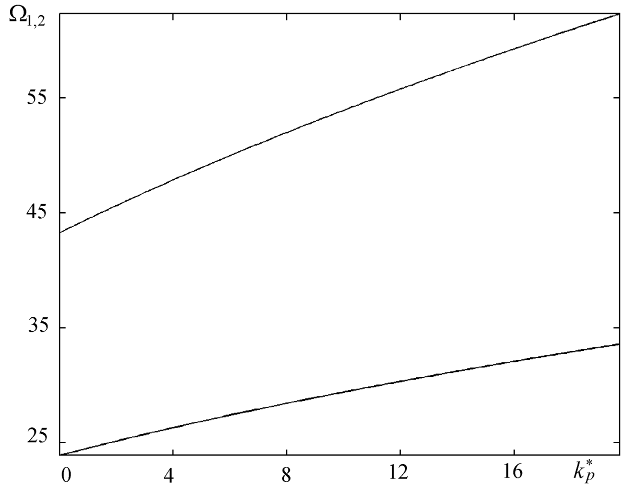

Fig. 7. The lowest two natural frequencies of a clamped–clamped beam versus the second foundation parameter; $L_2/L_0 = 0.6$, $h_1/L_0 = 1/120$.

6. CONCLUSIONS

Two analytical models for vibrations of delaminated beams resting on two-parameter elastic foundation were developed. The methods were employed for analysis of free vibrations of homogeneous and composite beams with general boundary conditions. The influence of the delamination size and location on the first frequencies as well as mode shapes were investigated. The first frequencies of the beam with elastic foundation are relatively higher than the frequencies of the delaminated beam without foundation. The calculated frequencies based on the present models agree well with the published analytical and numerical data.

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Delaminatsiooni mõjust Pasternaki alusel asuvate komposiitmaterjalist talade võnkumisele

Helle Hein

On vaadeldud homogeensete ja komposiitmaterjalist kihiliste talade võnkumist delaminatsiooni korral. Töö eesmärgiks on välja töötada analüütiline mudel delaminatsiooniga talade võnkumise uurimiseks juhul, kui tala asetseb kaheparameetrilisel elastsel alusel. On esitatud kaks mudelit, saadud võrrandisüsteemid on lahendatud numbriliselt. On analüüsitud elastse aluse ja delaminatsiooni mõju võnkumise resonantssagedustele.